

QQQ – Further Statistics 1 – Chapter 7 – Probability Generating Functions (Version 2)

Total Marks: 26

(26 = Platinum, 23 = Gold, 21 = Silver, 18 = Bronze)

1. The discrete random variable X has probability generating function

$$G_X(t) = k \ln \left(\frac{2}{2-t} \right)$$

where k is a constant.

- (a) Find the exact value of k (1)
- (b) Find the exact value of $\text{Var}(X)$ (7)
- (c) Find $P(X = 3)$ (4)

(Total 12 marks)

2. The probability generating function of the random variable X is

$$G_X(t) = k(1 + 2t)^5$$

where k is a constant.

(a) Show that $k = \frac{1}{243}$ (2)

(b) Find $P(X = 2)$ (2)

(c) Find the probability generating function of $W = 2X + 3$ (2)

The probability generating function of the random variable Y is

$$G_Y(t) = \frac{t(1 + 2t)^2}{9}$$

Given that X and Y are independent,

(d) find the probability generating function of $U = X + Y$ in its simplest form. (2)

(e) Use calculus to find the value of $\text{Var}(U)$ (6)

(Total 14 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$G(1) = 1 \Rightarrow k \ln 2 = 1 \text{ so } k = \underline{\underline{\frac{1}{\ln 2}}}$	B1	2.1
	(1)	

Notes

B1 for finding k (must be exact)

Mark scheme (b)

Scheme	Marks	AO
$\left\{ G(t) = \frac{1}{\ln 2} [\ln 2 - \ln(2 - t)] \right\} \Rightarrow G'(t) = \frac{1}{\ln 2} \left[\frac{1}{2 - t} \right]$ or $\frac{1}{\ln 2} (2 - t)^{-1}$	M1 A1	2.1 1.1b
$[E(X) =] G'(1) = \frac{1}{\ln 2}$	A1	1.1b
$G''(t) = \frac{1}{\ln 2} \times \left[\frac{1}{(2 - t)^2} \right]$	M1 A1	2.1 1.1b
$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2 = \frac{1}{\ln 2} + \frac{1}{\ln 2} - \left(\frac{1}{\ln 2} \right)^2$	M1	2.1
$= \underline{\underline{\frac{1}{\ln 2} \left(2 - \frac{1}{\ln 2} \right)}}$	A1	1.1b
	(7)	

Notes

1st M1 for an attempt to differentiate $G(t)$ e.g. $A(2 - t)^{-1}$ (o.e.)

1st A1 for a correct first derivative (condone k or use of $\frac{1}{\ln 2} = \text{awrt } 1.44$)

2nd A1 for correct $E(X)$ or $G'(1)$ (allow awrt 1.44 calc: 1.442695... but not k) seen anywhere

2nd M1 for attempting second derivative (ft their $G'(t)$)

3rd A1 for a correct 2nd derivative (condone k or use of $\frac{1}{\ln 2} = \text{awrt } 1.44$)

3rd M1 for a correct method for $\text{Var}(X)$ (some substitution into the correct formula)

4th A1 for $\frac{1}{\ln 2} \left(2 - \frac{1}{\ln 2}\right)$ o.e. but must simplify i.e. collect like terms

[Mark final answer – penalise incorrect log work etc]

NB 0.8040211... is A0 unless exact answer seen

Mark scheme (c)

Scheme	Marks	A0
$P(X = 3) = \text{coefficient of } t^3 \text{ by Maclaurin need } G'''(0)$	M1	3.1a
$G'''(t) = \frac{1}{\ln 2} \frac{2}{(2-t)^3}$	A1ft	1.1b
$P(X = 3) = \frac{G'''(0)}{3!}$	M1	3.2a
$= \frac{\frac{1}{4\ln 2}}{6} = \frac{1}{24 \ln 2} = 0.0601122... \text{ awrt } \underline{\underline{0.0601}}$	A1	1.1b
	(4)	
	(12 marks)	

Notes

1st M1 for a suitable strategy to solve the problem (finding link with Maclaurin)

Need mention of coefficient of t^3 **and** [$G'''(t)$ or $G'''(0)$] (condone $G'''(1)$)

1st A1ft for 3rd derivative, ft their 2nd derivative in (b) (provided $G''(t)$ not const)

Correct $G'''(t)$ or $G'''(0)$ scores 1st M1 1st A1ft

2nd M1 for translating Maclaurin to probability (a correct expression)

2nd A1 for $\frac{1}{24 \ln 2}$ or awrt 0.0601

ALT (c) Log series

1st M1 attempt to write $G(t)$ in suitable form as far as: $k [\ln 2 - \ln (2 [1 - \frac{t}{2}])]$

1st A1 reaching $-k \ln (1 - \frac{t}{2})$

2nd M1 use of $-\ln(1-x)$ series (some correct substitution) NB $G(t) = \frac{1}{\ln 2} \left(\frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{24} + \dots \right)$

2.

Mark scheme (a)

Scheme	Marks	AO
$G_X(1) = 1$	M1	2.1
$k \times 3^5 = 1 \quad \therefore k = \frac{1}{243}^*$	A1*cso	1.1b
	(2)	

Notes

M1: Stating $G_X(1) = 1$ eg $G_X(1) = k(1+2)^5 = 1 \quad k(1+2)^5 = 1$

Allow Verification $\frac{1}{243} \times 3^5 = 1$

A1*: Fully correct proof with no errors Substituting $t = 1$

Verification need therefore $G_X(1) = 1$

Mark scheme (b)

Scheme	Marks	AO
$P(X = 2)$ is coefficient of t^2 so $G_X(t) = k(\dots + {}^5C_2(2t)^2 + \dots)$	M1	1.1b
$P(X = 2) = \frac{40}{243}$	A1	1.1b
	(2)	

Notes

M1: Attempting to find the coefficient of t^2

A1: $\frac{40}{243}$ or awrt 0.165

Mark scheme (c)

Scheme	Marks	AO
$G_W(t) = \frac{t^3}{243} (1 + 2(t^2))^5$	M1	3.1a

$G_W(t) = \frac{t^3}{243} (1 + 2t^2)^5$	A1	1.1b
	(2)	

Notes

M1: Realising the need to multiply through by t^3 or subst t^2 for t

A1: $\frac{t^3}{243} (1 + 2t^2)^5$ oe eg $\frac{t^3}{243} (1 + 10t^2 + 40t^4 + 80t^6 + 80t^8 + 32t^{10})$

Mark scheme (d)

Scheme	Marks	AO
$G_U(t) = \frac{1}{243} (1 + 2t)^5 \times \frac{t(1 + 2t)^2}{9}$	M1	3.1a
$= \frac{t(1 + 2t)^7}{2187}$	A1	1.1b
	(2)	

Notes

M1: Realising the need to use $G_U(t) = G_X(t) \times G_Y(t)$

A1: $\frac{t(1 + 2t)^7}{2187}$ oe

Mark scheme (e)

Scheme	Marks	AO
$G_U'(t) = \frac{14t(1 + 2t)^6}{2187} + \frac{(1 + 2t)^7}{2187}$	M1	2.1
$G_U'(1) = \frac{17}{3}$	A1ft	1.1b
$G_U''(t) = \frac{168t(1 + 2t)^5}{2187} + \frac{14(1 + 2t)^6}{2187} + \frac{14(1 + 2t)^6}{2187}$	M1	2.1
$G_U''(1) = 28$	A1	1.1b
$\text{Var}(U) = "28" + "\frac{17}{3}" - \left("\frac{17}{3}" \right)^2$	M1	2.1
$= \frac{14}{9}$	A1	1.1b

	(6)	
	(14 marks)	

Notes

M1: For an attempt to differentiate $G(u)$ e.g. $G_U'(t) = At(1+2t)^6 + B(1+2t)^7$ ft their part(d) if in the form $kt(1+2t)^n$ where $n \geq 5$

A1ft: $\frac{17}{3}$ or awrt 5.67

M1: For attempting second derivative eg $G_U''(t) = Ct(1+2t)^5 + D(1+2t)^6$ ft their part(d) if in the form $kt(1+2t)^n$ where $n \geq 5$

A1 28

M1: Using $G_U''(1) + G_U'(1) - (G_U'(1))^2$ ft their values

A1: $\frac{14}{9}$ or awrt 1.56

Alternative (e)

Scheme	Marks
$G_X''(t) = A(1+2t)^3$	M1
$G_X'(1) = \frac{10}{3}$ and $G_X''(1) = \frac{80}{9}$	A1ft
$G_Y''(t) = H(8+24t)$	M1
$G_Y'(1) = \frac{7}{3}$ and $G_Y''(1) = \frac{32}{9}$	A1
Using $G_U''(1) + G_U'(1) - (G_U'(1))^2$ to find $\text{Var}(X)$, $\text{Var } Y$ and $\text{Var } U$	M1
$\frac{14}{9}$ or awrt 1.56	A1