

QQQ – Further Statistics 1 – Chapter 7 – Probability Generating Functions (Version 1)

Total Marks: 24

(24 = Platinum, 22 = Gold, 19 = Silver, 16 = Bronze)

1. The discrete random variable X has probability generating function

$$G_X(t) = \frac{1}{\left(1 - \frac{2t}{5}\right)} - k$$

where k is a constant.

(a) Find the exact value of k (2)

(b) Find the exact value of $E(X)$ (3)

(c) Find $P(X = 2)$ (3)

The random variable Y is defined as $Y = 4X - 1$

(d) Find the probability generating function of Y (2)

(Total 10 marks)

2. The discrete random variable V has probability distribution

v	2	3	4
$P(V = v)$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

(a) Show that the probability generating function of V is

$$G_V(t) = t^2 \left(\frac{2}{5}t + \frac{3}{5} \right)^2$$

(2)

The discrete random variable W has probability generating function

$$G_W(t) = t \left(\frac{2}{5}t + \frac{3}{5} \right)^5$$

(b) Use calculus to find

(i) $E(W)$ **(2)**

(ii) $\text{Var}(W)$ **(4)**

Given that V and W are independent,

(c) find the probability generating function of $X = V + W$ in its simplest form. **(2)**

The discrete random variable $Y = 2X + 3$

(d) Find the probability generating function of Y **(2)**

(e) Find $P(Y = 15)$ **(2)**

(Total 14 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$G_X(1) = 1 \Rightarrow \frac{1}{1 - \frac{2}{5}} - k = 1$	M1	2.1
$k = \frac{2}{3}$	A1	1.1b
	(2)	

Notes

M1: For writing or using $G_X(1) = 1$

A1: For finding $k = \frac{2}{3}$ (must be exact)

Mark scheme (b)

Scheme	Marks	AO
$G'_X(t) = -\frac{-\frac{2}{5}}{\left(1 - \frac{2t}{5}\right)^2}$	M1	2.1
$[G'_X(t) =] \frac{2}{5} \left(1 - \frac{2t}{5}\right)^{-2}$ may see $[G'_X(t) =] \frac{10}{(5 - 2t)^2}$	A1	1.1b
$G'_X(1) = \frac{2}{5 \left(1 - \frac{2}{5}\right)^2} \Rightarrow [E(X) =] \frac{10}{9}$	A1	1.1b
	(3)	

Notes

M1: Attempt to differentiate $G_X(t)$ to obtain $A \left(1 - \frac{2t}{5}\right)^{-2}$ (o.e.) for some constant A

A1: Correct differential (may be unsimplified)

A1: For finding $[E(X) =] \frac{10}{9}$ (must be exact, and must come from a correct differential)

Mark scheme (c)

Scheme	Marks	AO
$G_X''(t) = (-2) \times \frac{2 \times \left(-\frac{2}{5}\right)}{5 \left(1 - \frac{2t}{5}\right)^3} = \frac{8}{25 \left(1 - \frac{2t}{5}\right)^3}$ or Use of binomial expansion or standard power series (at least two terms or t^2 term correct)	M1	3.1a
Use of Maclaurin series for t^2 (use of $P(X = 2) = \frac{G_X''(0)}{2!}$) or $\frac{(-1)(-2) \left(-\frac{2t}{5}\right)^2}{2!}$	M1	1.1b
$\frac{1}{2} \times \frac{8}{25} = \frac{4}{25}$ or $\frac{4}{25}$	A1	1.1b
	(3)	

Notes

M1: For $G_X''(t)$ in the form $A \left(1 - \frac{2t}{5}\right)^{-3}$ **or** a correct expression (not value) for $G_X''(0)$

Series method: Two terms correct from: $\frac{1}{3} + (-1) \left(-\frac{2}{5}t\right) + \frac{(-1)(-2)}{2} \left(-\frac{2}{5}t\right)^2$

M1: For correct method to find $G_X''(t)$ and use of $P(X = 2) = \frac{G_X''(0)}{2!}$ (ft their $G_X'(t)$ in (b) provided $G_X'(t)$ not constant)

Series method: Correct unsimplified expression for t^2 seen (may be part of expansion)

A1: For $\frac{4}{25}$ o.e.

Mark scheme (d)

Scheme	Marks	AO
Use of $\frac{1}{t}G_X(t)$ or $G_X(t^4)$ shown	M1	3.1a
$G_Y(t) = \frac{1}{t} \left(\frac{1}{1 - \frac{2t^4}{5}} - \frac{2}{3} \right)$	A1ft	1.1b
	(2)	
	(10 marks)	

Notes

M1: Sight or use of $\frac{1}{t}G_X(t)$ or $G_X(t^4)$ stated or effected in expression

A1ft: Correct expression, ft their k (accept k itself). ISW following a correct expression.

2.

Mark scheme (a)

Scheme	Marks	AO
$G_V(t) = \frac{9}{25}t^2 + \frac{12}{25}t^3 + \frac{4}{25}t^4$ <u>or</u> $t^2 \left(\frac{9}{25} + \frac{12}{25}t + \frac{4}{25}t^2 \right)$	M1	1.1b
$= t^2 \left(\frac{2}{5}t + \frac{3}{5} \right)^2$ *	A1* cso	2.1
	(2)	

Notes

M1 A correct un-simplified pgf based on $\sum t^v P(V = v)$

A1* cso must see an un-simplified version i.e. M1 scored and no incorrect working seen

Mark scheme (b)

Scheme	Marks	AO
(i) $G_W'(t) = 2t \left(\frac{2}{5}t + \frac{3}{5} \right)^4 + \left(\frac{2}{5}t + \frac{3}{5} \right)^5$	M1	2.1
$[G_W'(1) =] \underline{\mathbf{3}}$	A1	1.1b
(ii) $G_W''(t) = 2 \left(\frac{2}{5}t + \frac{3}{5} \right)^4 + \frac{16}{5}t \left(\frac{2}{5}t + \frac{3}{5} \right)^3 + 2 \left(\frac{2}{5}t + \frac{3}{5} \right)^4$ oe	M1	2.1
$G_W''(1) = \frac{36}{5}$	A1	1.1b
$\text{Var}(W) = \frac{36}{5} + \text{“3”} - (\text{“3”})^2$	M1	2.1
$= \frac{6}{5}$	A1	1.1b
	(6)	

Notes

(i) **M1** Differentiating using the product rule to find $G_W'(t)$ Allow un-simplified e.g. $5 \times \frac{2}{5}t$

Need two terms added and at least one correct. If they expand we need 3 correct.

A1 3 from a correct derivative

(ii) **1st M1** Attempt $G_W''(t)$ ft their $G_W'(t)$ [must be at least 2 terms or a product], one correct ft term, same rule for differentiating a product

1st A1 $\frac{36}{5}$ or 7.2 from a correct derivative

2nd M1 $G_W''(1) + G_W'(1) - (G_W'(1))^2$ ft their $G_W''(t)$ if different from $G_W'(t)$ and $G_W(t)$

2nd A1 Dep on **M3A2** $\frac{6}{5}$ or 1.2

Alternative for (b)

Scheme	Marks	AO
$W = P + 1$ where $P \sim B(5, 0.4)$ so $\text{Var}(W) = \text{Var}(P)$		
(i) $G_P'(t) = 2 \left(\frac{2}{5}t + \frac{3}{5} \right)^4$	M1	2.1
$G_W'(1) = 2 + 1 = 3$	A1	1.1b
(ii) $G_P''(t) = \frac{16}{5} \left(\frac{2}{5}t + \frac{3}{5} \right)^3$; $G_P''(1) = \frac{16}{5}$	M1; A1	2.1 1.1b
$\text{Var}(W) = \frac{16}{5} + 2 - (2)^2$; $= \frac{6}{5}$	M1; A1	2.1 1.1b

SC MR They use $G_V(t)$ instead of $G_W(t)$ Provided some correct differentiation seen:

Award B1 for $E(V) = \frac{14}{5}$ and B1 for $\text{Var}(V) = \frac{12}{25}$ score as M0A1M0A0M0A1

Mark scheme (c)

Scheme	Marks	AO
$G_X(t) = t^2 \left(\frac{2}{5}t + \frac{3}{5} \right)^2 \times t \left(\frac{2}{5}t + \frac{3}{5} \right)^5$	M1	3.1a
$= t^3 \left(\frac{2}{5}t + \frac{3}{5} \right)^7$	A1	1.1b
	(2)	

Notes

M1 Realising the need to use $G_X(t) = G_V(t) \times G_W(t)$

A1 $t^3 \left(\frac{2}{5}t + \frac{3}{5} \right)^7$

Mark scheme (d)

Scheme	Marks	AO
$G_Y(t) = t^3 \times (t^2)^3 \times \left(\frac{2}{5}t^2 + \frac{3}{5} \right)^7$	M1	3.1a
$= t^9 \left(\frac{2}{5}t^2 + \frac{3}{5} \right)^7$	A1	1.1b
	(2)	

Notes

M1 Realising the need to multiply through by t^3 or substitute t^2 for t or sight of $t^3 G_X(t^2)$

A1 $t^9 \left(\frac{2}{5}t^2 + \frac{3}{5} \right)^7$ oe Need not be in its simplest form

Mark scheme (e)

Scheme	Marks	AO
$P(Y = 15)$ is coefficient of t^{15} ie $\dots + t^9 \times {}^7C_3 \left(\frac{2}{5}t^2 \right)^3 \left(\frac{3}{5} \right)^4 + \dots$ <u>or</u> $P(X = 6)$ need coefficient of t^6 i.e. $\dots + t^3 \times {}^7C_3 \left(\frac{2}{5}t \right)^3 \left(\frac{3}{5} \right)^4 + \dots$	M1	1.1b
$[P(Y = 15) =] \frac{22680}{78125} = \frac{4536}{15625} = 0.290304$	A1	1.1b
	(2)	
	(14 marks)	

Notes

M1 Attempting to find correct coefficient of t^n or identify $Y = 2J + 9$ where $J \sim B(7, 0.4)$

Need an expression can fit their $G_Y(t)$ or $G_X(t)$ of the form $t^n(at^m + b)^k$

Allow a statement that $P(Y = 15) = 0$ if it follows from their pgf

A1 For a correct exact answer or allow awrt 0.2903 Allow 0.29 from correct expression