

QQQ – Further Pure 2 – Chapter 6 – Integration Techniques (Version 2)

Total Marks: 22

(22 = Platinum, 20 = Gold, 18 = Silver, 15 = Bronze)

1. The curve C has equation

$$y = \frac{x^2}{8} - \ln x, \quad 2 \leq x \leq 3$$

Find the length of the curve C giving your answer in the form $p + \ln q$, where p and q are rational numbers to be found.

(7)

(Total 7 marks)

Edexcel FP3 June 2016 Q2

2.

$$I_n = \int (4 - x^2)^{-n} dx \quad n > 0$$

(a) Show that, for $n > 0$

$$I_{n+1} = \frac{x}{8n(4 - x^2)^n} + \frac{2n - 1}{8n} I_n$$

(5)

(b) Find I_2

(3)

(Total 8 marks)

Edexcel A2 October 2020 Q7

3. The curve with parametric equations

$$x = \cosh 2\theta, \quad y = 4 \sinh \theta, \quad 0 \leq \theta \leq 1$$

is rotated through 2π radians about the x -axis.

Show that the area of the surface generated is $\lambda(\cosh^3 \alpha - 1)$, where $\alpha = 1$ and λ is a constant to be found.

(7)

(Total 7 marks)

Edexcel FP3 June 2013 Q3

Solutions

1.

Scheme	Marks
$y = \frac{x^2}{8} - \ln x, \quad 2 \leq x \leq 3$	
$\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}$	B1
$L = \int \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} dx = \int \sqrt{\left(1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2\right)} dx$	M1
$= \int \sqrt{\left(1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}\right)} dx = \int \sqrt{\left(\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}\right)} dx =$ $\int \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2} dx = \int \left(\frac{x}{4} + \frac{1}{x}\right) dx$	M1 A1
$= \frac{x^2}{8} + \ln kx$	A1
$\left[\frac{x^2}{8} + \ln x\right]_2^3 = \left(\frac{3^2}{8} + \ln 3\right) - \left(\frac{2^2}{8} + \ln 2\right)$	M1
$\frac{5}{8} + \ln \frac{3}{2}$	A1
	(7)
	(7 marks)

Notes

B1: Correct derivative. Allow any correct equivalent e.g. $\frac{2x}{8} - \frac{1}{x}$

M1: Use of a correct formula using their derivative and not the given y .

M1: Squares their derivative to obtain $ax^2 + bx^{-2} + c$, where none of a , b or c are zero – this may be implied by e.g. $\frac{ax^4 + bx^2 + c}{dx^2}$ **and** adds 1 to their constant term.

A1: Correct integrand $\frac{x}{4} + \frac{1}{x}$ or equivalent e.g. $\frac{x^2 + 4}{4x}$ (integral sign not needed)

A1: Correct integration

M1: Substitutes 2 and 3 into an expression of the form $px^2 + q \ln x$ ($p, q \neq 0$) and subtracts the right way round. Must be seen explicitly or may be implied by a correct exact answer for their integration. If the candidate gives the **final single answer** in decimals with no substitution shown, e.g. 1.030...this is M0.

A1: Cao and cso (oe e.g $0.625 + \ln \frac{3}{2}$)

2.

Mark scheme (a)

Scheme	Marks	AO
$I_n = \int 1 \times (4 - x^2)^{-n} dx = x(4 - x^2)^{-n} \pm \int x.n(4 - x^2)^{-n-1}.2x dx$	M1	3.1a
$I_n = x(4 - x^2)^{-n} - \int x.-n(4 - x^2)^{-n-1}.2x dx$	A1	1.1b
$= x(4 - x^2)^{-n} + 2n \int (4 - x^2 - 4)(4 - x^2)^{-n-1} dx$ $= x(4 - x^2)^{-n} + 2n \int (4 - x^2)(4 - x^2)^{-n-1} - 4(4 - x^2)^{-n-1} dx$	M1	2.1
$I_n = x(4 - x^2)^{-n} + 2n \int (4 - x^2)^{-n} dx - 8n \int (4 - x^2)^{-(n+1)} dx$ $= x(4 - x^2)^{-n} + 2nI_n - 8nI_{n+1}$	M1	1.1b
$\Rightarrow 8nI_{n+1} = x(4 - x^2)^{-n} + (2n - 1)I_n \Rightarrow I_{n+1} =$ $\frac{x}{8n(4 - x^2)^n} + \frac{2n - 1}{8n}I_n$ *	A1*	2.1
	(5)	

Notes

M1: Splits integrand as $1 \times (4 - x^2)^{-n}$ and attempts parts (the right way round). There are lots of minus signs around, so accept with $\pm$ between as it is hard to tell if the formula is incorrect or it is sign error.

A1: Correct result obtained, need not be simplified.

M1: Gathers the terms and writes the x^2 as $-(4 - x^2 - 4)$ (oe). Accept either line shown in the scheme (with their coefficients from the parts expansion).

M1: Sorts out the indices and replaces the appropriate integrals by I_n and I_{n+1} respectively.

A1*: Completes to the correct printed result, with no errors or ambiguities.

Mark scheme (b)

Scheme	Marks	AO
$I_1 = \int \frac{1}{4-x^2} dx = \frac{1}{2} \operatorname{artanh} \left(\frac{x}{2} \right) \text{ or } \frac{1}{4} \ln \left \frac{2+x}{2-x} \right \text{ oe}$	B1	2.2a
$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{8} I_1$	M1	1.1b
$= \frac{x}{8(4-x^2)} + \frac{1}{16} \operatorname{artanh} \left(\frac{x}{2} \right) (+c) \text{ oe e.g.}$ $\frac{x}{8(4-x^2)} + \frac{1}{32} \ln \left \frac{2+x}{2-x} \right (+c)$	A1	1.1b
	(3)	
	(8 marks)	

Notes

B1: Deduces the correct result for I_1

M1: Applies the reduction formula correct with $n = 1$ only.

A1: Correct answer, accept any equivalent form as long as fractions are simplified. Constant of integration is not needed.

3.

Scheme	Marks
$\left(\frac{dx}{d\theta} \right) = 2 \sinh 2\theta \text{ and } \left(\frac{dy}{d\theta} \right) = 4 \cosh \theta$ <i>Or equivalent correct derivatives</i>	B1
$A = (2\pi) \int 4 \sinh \theta \sqrt{"2 \sinh 2\theta"}^2 + {"4 \cosh \theta"}^2 d\theta$ or $A = (2\pi) \int 4 \sinh \theta \sqrt{\left(1 + \left(\frac{"4 \cosh \theta"}{"2 \sinh 2\theta"} \right)^2 \right)} \cdot 2 \sinh 2\theta d\theta$ <i>Use of correct formula including replacing dx with $"2 \sinh 2\theta" d\theta$ if chain rule used. Allow the omission of the 2π here.</i>	M1

$A = 32\pi \int \sinh \theta \cosh^2 \theta \, d\theta$ $A = 32\pi \int (\sinh \theta + \sinh^3 \theta) \, d\theta$ <u>Completely correct expression for A with the square root removed</u> <i>This mark may be recovered later if the 2π is introduced later</i>	B1
$A = \frac{32\pi}{3} [\cosh^3 \theta]_0^1$ M1: Valid attempt to integrate a correct expression or a multiple of a correct expression – dependent on the first M1 A1: Correct expression	dM1A1
$= \frac{32\pi}{3} [\cosh^3 1 - 1]$ M1: Uses the limits 0 and 1 correctly. Dependent on both previous M's A1: Cao and cso (no errors seen)	ddM1A1
	(7)
	(7 marks)

Notes

Example Alternative Integration for last 4 marks

Scheme	Marks
$\int \sinh \theta \cosh^2 \theta \, d\theta = \int \sinh \theta (1 + \sinh^2 \theta) \, d\theta = \int (\sinh \theta + \sinh^3 \theta) \, d\theta$ $\int \left(\sinh \theta + \frac{1}{4} \sinh 3\theta - \frac{3}{4} \sinh \theta \right) \, d\theta = \frac{1}{4} \int (\sinh \theta + \sinh 3\theta) \, d\theta$ $= \frac{1}{4} \cosh \theta + \frac{1}{12} \cosh 3\theta$ dM1: $\int \sinh \theta \cosh^2 \theta \, d\theta = p \cosh \theta + q \cosh 3\theta$ A1: $32\pi \left[\frac{1}{4} \cosh \theta + \frac{1}{12} \cosh 3\theta \right]$	dM1A1
$A = 8\pi \left[\cosh \theta + \frac{1}{3} \cosh 3\theta \right]_0^1$ $= 8\pi \left(\cosh 1 + \frac{1}{3} \cosh 3 - \cosh 0 - \frac{1}{3} \cosh 0 \right)$ $\frac{32\pi}{3} [\cosh^3 1 - 1]$	ddM1A1

M1: Uses the limits 0 and 1 correctly. Dependent on **both** previous M's

A1: Cao

Alternative Cartesian Approach

Scheme	Marks
$x = 1 + \frac{y^2}{8}$ Any correct Cartesian equation	B1
$\frac{dx}{dy} = \frac{y}{4} \text{ or } \frac{dy}{dx} = \frac{\sqrt{2}}{(x-1)^{\frac{1}{2}}}$ Correct Derivative	B1
$A = \int 2\pi \cdot y \sqrt{\left(1 + \left(\frac{y}{4}\right)^2\right)} dy \text{ or }$ $A = \int 2\pi \cdot \sqrt{8}(x-1)^{\frac{1}{2}} \sqrt{\left(1 + \left(\frac{2}{x-1}\right)\right)} dx$ Use of a correct formula	M1
$A = 2\pi \times \frac{2}{3} \times 8 \left(1 + \frac{y^2}{16}\right)^{\frac{3}{2}} \text{ or } A = \frac{4\pi\sqrt{8}}{3}(x+1)^{\frac{3}{2}}$ M1: Convincing attempt to integrate a relevant expression – dependent on the first M1 but allow the omission of 2π A1: Completely correct expression for A	dM1 A1
$A = 2\pi \times \frac{2}{3} \times 8(1 + \sinh^2 1)^{\frac{3}{2}} - 2\pi \times \frac{2}{3} \times 8 \text{ or }$ $2\pi \times \frac{2}{3} \times \sqrt{8}(1 + \cosh 2)^{\frac{3}{2}} - \frac{32\pi}{3}$ Correct use of limits ($0 \rightarrow 4 \sinh 1$ for y or $1 \rightarrow \cosh 2$ for x)	ddM1
Use $1 + \sinh^2 1 = \cosh^2 1$ to give $\frac{32\pi}{3} [\cosh^3 1 - 1]$ or Use $\cosh 2 = 2 \cosh^2 1 - 1$ to give $\frac{32\pi}{3} [\cosh^3 1 - 1]$	A1