

QQQ – Further Pure 2 – Chapter 5 – Matrix Algebra (Version 2)

Total Marks: 22

(22 = Platinum, 20 = Gold, 18 = Silver, 15 = Bronze)

1.

$$\mathbf{A} = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$$

where a is a constant.

(a) Determine, in expanded form in terms of a , the characteristic equation for $\mathbf{A}$. (2)

(b) Hence use the Cayley-Hamilton theorem to determine values of a and b such that

$$\mathbf{A}^3 = \mathbf{A} + b\mathbf{I}$$

where $\mathbf{I}$ is the 2×2 identity matrix. (4)

(Total 6 marks)

Edexcel A2 June 2023 Q1

2.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$\mathbf{A} = \begin{pmatrix} 3 & k \\ -5 & 2 \end{pmatrix}$$

where k is a constant.

Given that there exists a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix where

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$$

(a) show that $k = -6$ (3)

(b) determine a suitable matrix $\mathbf{P}$ (4)

(Total 7 marks)

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3.

$$\mathbf{M} = \begin{pmatrix} 3 & k & 2 \\ -1 & 0 & 1 \\ 1 & k & 1 \end{pmatrix}, \quad \text{where } k \text{ is a constant}$$

Given that 3 is an eigenvalue of $\mathbf{M}$,

(a) find the value of k . **(3)**

(b) Hence find the other two eigenvalues of $\mathbf{M}$. **(4)**

(c) Find an eigenvector corresponding to the eigenvalue 3 **(2)**

(Total 9 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$\det \begin{pmatrix} -1-\lambda & a \\ 3 & 8-\lambda \end{pmatrix} = 0 \Rightarrow (-1-\lambda)(8-\lambda) - 3a = 0 \Rightarrow \dots$	M1	1.1b
$\lambda^2 - 7\lambda - 3a - 8 = 0$ o.e.	A1	1.1b
	(2)	

Notes

M1: Correct method to find the characteristic equation for **A**, condone missing = 0

A1: Correct simplified characteristic equation.

Mark scheme (b)

Scheme	Marks	AO
$\mathbf{A}^2 - 7\mathbf{A} - 3a\mathbf{I} - 8\mathbf{I} = 0 \Rightarrow \mathbf{A}^3 = 7\mathbf{A}^2 + (3a + 8)\mathbf{A}$	M1	1.1b
$\Rightarrow \mathbf{A}^3 = 7(7\mathbf{A} + (3a + 8)\mathbf{I}) + (3a + 8)\mathbf{A}$ Or $\mathbf{A}^3 = 7 \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}^2 + (8 + 3a) \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$ $= 7 \begin{pmatrix} 1 + 3a & 7a \\ 21 & 3a + 64 \end{pmatrix} + \begin{pmatrix} -8 - 3a & 8a + 3a^2 \\ 24 + 9a & 64 + 24a \end{pmatrix}$	M1	2.1
$\Rightarrow \mathbf{A}^3 = (3a + 8 + 49)\mathbf{A} + 7(3a + 8)\mathbf{I} \Rightarrow 3a + 57 = 1 \Rightarrow a = \dots$ Or $\begin{pmatrix} b-1 & a \\ 3 & b+8 \end{pmatrix} = \begin{pmatrix} 18a-1 & 3a^2+57a \\ 171+9a & 512+45a \end{pmatrix} \Rightarrow \text{e.g.}$ $3 = 171 + 9a \Rightarrow a = \dots$ Or	M1	1.1b

$\mathbf{A}^3 = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix} + \begin{pmatrix} 18a & 3a^2 + 56a \\ 168 + 9a & 504 + 45a \end{pmatrix} \Rightarrow \text{e.g.}$ $18a = 504 + 45a \Rightarrow a = \dots$		
$\Rightarrow a = -\frac{56}{3}, b = -336$	A1	1.1b
	(4)	
	(6 marks)	

Notes

Note: this question asks the candidates to use the Cayley-Hamilton theorem so any other approach that doesn't score the first method mark scores no marks

M1: Uses the Cayley-Hamilton theorem with their equation and multiplies though by $\mathbf{A}$ to find an equation for $\mathbf{A}^3$

M1: Substitutes for $\mathbf{A}^2$ to obtain an equation for $\mathbf{A}^3$ in terms of a , $\mathbf{I}$ and $\mathbf{A}$. Alternatively substitutes in the matrix $\mathbf{A}$ and attempts to square

M1: Equates coefficient(s) of $\mathbf{A}$ to 1 and proceeds to find a value for a . Alternatively equates elements to find a value for a .

A1: Correct values for a and b .

Special case: Missing matrix $\mathbf{I}$ from their working can score maximum of M1 (if multiply by $\mathbf{A}$ correctly) M1M1A0 unless implied $\mathbf{I}$ from their working this can score all marks.

(corrected from the printed mark scheme: in the two "Or" methods for the third mark the bottom-right entries are printed as $512 + 27a$ and $504 + 27a$, and the example as $18a = 504 + 27a$; the correct entries are $512 + 45a$ and $504 + 45a$)

2.

Mark scheme (a)

Scheme	Marks	A0
Uses entries on the diagonal matrix so eigenvalues are -3 and 8	B1	1.2
$\begin{vmatrix} 3 - \lambda & k \\ -5 & 2 - \lambda \end{vmatrix} = 0 \Rightarrow (3 - \lambda)(2 - \lambda) + 5k = 0$ $\Rightarrow \lambda^2 - 5\lambda + 6 + 5k = 0$ $\Rightarrow 6 + 5k = -24 \Rightarrow k = \dots$	M1	3.1a
$k = -6$	A1	1.1b

Notes

B1: Recalls that the eigenvalues form the entries of the diagonal matrix in the diagonalisation process. May be implied by working or stated, for work in (a) or (b). Implied by an attempt at equating the determinants of the two matrices.

M1: For a full method to find k , e.g. by finding the characteristic equation and using (one of) the eigenvalues to find k . The factor theorem can be used with either eigenvalue, for example, after finding the characteristic equation.

A1: Correct value.

Note Using $k = -6$

B1: Recalls that the eigenvalues form the entries of the diagonal matrix in the diagonalisation process.

M1: Uses $k = -6$, finds the characteristic equation and solves to find the eigenvalues.

A1: Draws a conclusion, same eigenvalues therefore $k = -6$

Mark scheme (b)

Scheme	Marks	AO
For $\lambda = -3$ eigenvector equations are $\begin{cases} 6x - "6"y = 0 \\ -5x + 5y = 0 \end{cases} \Rightarrow x, y = \dots$ OR For $\lambda = 8$ eigenvector equations are $\begin{cases} -5x - "6"y = 0 \\ -5x - 6y = 0 \end{cases} \Rightarrow x, y = \dots$	M1	2.1
(For $\lambda = -3, x = y$, for $\lambda = 8, 5x = -6y$ so eigenvectors are) One of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ o.e. and $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$ o.e.	A1	1.1b
Both of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ o.e. and $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$ o.e.	A1	1.1b
$\mathbf{P} = \begin{pmatrix} 6 & 1 \\ -5 & 1 \end{pmatrix}$ o.e.	B1ft	2.2a
	(4)	
	(7 marks)	

Notes

M1: Correct method to find an eigenvector for either of their eigenvalues.

A1: One correct eigenvector – accept any non-zero multiples of their eigenvalues.

A1: A correct eigenvector for each of their eigenvalues.

B1ft: Gives **P** as a matrix with their columns in the correct order for the given diagonal matrix.

3.

Mark scheme (a)

Scheme	Marks
$\mathbf{M} = \begin{pmatrix} 3 & k & 2 \\ -1 & 0 & 1 \\ 1 & k & 1 \end{pmatrix}$	
$\begin{vmatrix} 3-3 & k & 2 \\ -1 & -3 & 1 \\ 1 & k & 1-3 \end{vmatrix} = (3-3)[-3(1-3)-k] - k[-1(1-3)-1] + 2(-k+3) (= -3k+6)$ Attempts determinant of $\mathbf{M} - 3\mathbf{I}$ or $\begin{vmatrix} 3-\lambda & k & 2 \\ -1 & -\lambda & 1 \\ 1 & k & 1-\lambda \end{vmatrix} = (3-\lambda)[- \lambda(1-\lambda)-k] - k[-1(1-\lambda)-1] + 2(-k+\lambda)$ $(-\lambda^3 + 4\lambda^2 - \lambda - 3k)$ $\lambda = 3 \Rightarrow \det(\mathbf{M} - \lambda\mathbf{I}) = (3-3)[-3(1-3)-k] - k[-1(1-3)-1] + 2(-k+3)$ Attempts determinant of $\mathbf{M} - \lambda\mathbf{I}$ and substitutes $\lambda = 3$	M1
$0 - k + 2(-k+3) = 0 \Rightarrow k = 2$	dM1 A1
	(3)

Notes

Should be a recognisable attempt at the determinant (if there is any doubt at least 2 “terms” should be correct)

dM1: Puts = 0 (may be implied) and solves for k . **Dependent on the first M**

A1: $k = 2$

Alternative to part (a)

Scheme	Marks
$\begin{pmatrix} 3 & k & 2 \\ -1 & 0 & 1 \\ 1 & k & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$	
$3x + ky + 2z = 3x$ $-x + z = 3y$ $x + ky + z = 3z$	M1
Let e.g. $x = 1$ $\Rightarrow ky = -2z, z = 3y + 1, ky + 1 = 2z$ $\Rightarrow 4z = 1 \Rightarrow z = \frac{1}{4}, y = -\frac{1}{4}$ $\Rightarrow k = \dots$ Or e.g. $\Rightarrow ky = -2z, ky = 2z - x, z = 3y + x$ $\Rightarrow ky = -6y - 2x, 2ky = 12y + 2x \Rightarrow 3ky = 6y$ $\Rightarrow k = \dots$	dM1
$\Rightarrow k = 2$	A1
	(3)

M1: Forms 3 equations using the eigenvalue 3

dM1: Allocates a non-zero value to one of x or y or z and solves for the other two variables and finds a value for k Or Solves to obtain a value for k **Dependent on the first M**

Mark scheme (b)

Scheme	Marks
$\det(\mathbf{M} - \lambda \mathbf{I}) = (3 - \lambda) [-\lambda(1 - \lambda) - k] - k [-1(1 - \lambda) - 1] + 2(-k + \lambda)$ Attempts determinant of $\mathbf{M} - \lambda \mathbf{I}$ (may be seen in (a)) but must be seen or used in (b) to score in (b)	M1
$\det(\mathbf{M} - \lambda \mathbf{I}) = (3 - \lambda) [-\lambda(1 - \lambda) - 2] - 2 [-1(1 - \lambda) - 1] + 2(-2 + \lambda) = 0$ Uses their k in their determinant and puts = 0 (May be implied by their work) Dependent on the first M	dM1
$\{(3 - \lambda)\} (\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = \dots$	ddM1

$\lambda = -1, 2$	A1
	(4)

Notes

ddM1: Solves 3TQ to find the 2 other eigenvalues (apply usual rules if necessary). **Dependent on both previous M's**

If they multiply out they should get $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$ and may use a calculator to obtain $\lambda = -1, 2$ (and 3)

A1: Correct eigenvalues. (Must follow $k = 2$)

Mark scheme (c)

Scheme	Marks
$\begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{aligned} 3x + 2y + 2z &= 3x \\ -x + z &= 3y \\ x + 2y + z &= 3z \end{aligned}$ or $\begin{pmatrix} 0 & 2 & 2 \\ -1 & -3 & 1 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 2y + 2z &= 0 \\ -x - 3y + z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$ Expands to obtain at least 2 equations. Allow if k is present.	M1
$k \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ or $k(4\mathbf{i} - \mathbf{j} + \mathbf{k})$	A1
	(2)
	(9 marks)

Notes

A1: Any non-zero multiple but must be a vector

****Note that the vector product of any 2 rows of $\mathbf{M} - 3\mathbf{I}$ also gives an eigenvector****

Note on Determinants

Note that determinants can be found using any row or column

And also by applying the rule of Sarrus which is:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Please look out for these alternative approaches in Question 3