

QQQ – Further Pure 2 – Chapter 4 – Recurrence Relations (Version 2)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1.

(a) Determine the general solution of the recurrence relation

$$u_n = 2u_{n-1} - u_{n-2} + 2^n \quad n \geq 2$$

(4)

(b) Hence solve this recurrence relation given that $u_0 = 2u_1$ and $u_4 = 3u_2$

(2)

(Total 6 marks)

2.

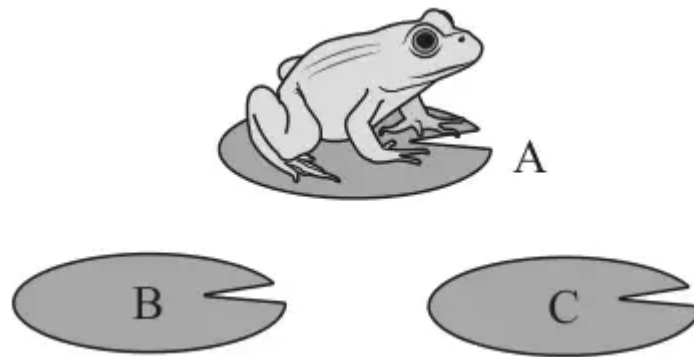


Figure 1

There are three lily pads on a pond. A frog hops repeatedly from one lily pad to another.

The frog starts on lily pad A, as shown in Figure 1.

In a model, the frog hops from its position on one lily pad to either of the other two lily pads with equal probability.

Let p_n be the probability that the frog is on lily pad A after n hops.

(a) Explain, with reference to the model, why $p_1 = 0$ (1)

The probability p_n satisfies the recurrence relation

$$p_{n+1} = \frac{1}{2}(1 - p_n) \quad n \geq 1 \quad \text{where } p_1 = 0$$

(b) Prove by induction that, for $n \geq 1$

$$p_n = \frac{2}{3} \left(-\frac{1}{2} \right)^n + \frac{1}{3}$$

(6)

(c) Use the result in part (b) to explain why, in the long term, the probability that the frog is on lily pad A is $\frac{1}{3}$ (1)

(Total 8 marks)

3. A student takes out a loan for £1000 from a bank.

The bank charges 0.5% monthly interest on the amount of the loan yet to be repaid.

At the end of each month

- the interest is added to the loan
- the student then repays £50

Let U_n be the amount of money owed n months after the loan was taken out.

The amount of money owed by the student is modelled by the recurrence relation

$$U_n = 1.005U_{n-1} - A \quad U_0 = 1000 \quad n \in \mathbb{Z}^+$$

where A is a constant.

- (a)** **(i)** State the value of the constant A .
- (ii)** Explain, in the context of the problem, the value 1.005

(2)

Using the value of A found in part (a)(i),

- (b)** solve the recurrence relation

$$U_n = 1.005U_{n-1} - A \quad U_0 = 1000 \quad n \in \mathbb{Z}^+$$

(5)

- (c)** Hence determine, according to the model, the number of months it will take to completely repay the loan.

(2)

(Total 9 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = \dots \{\lambda = 1\}$	M1	1.1b
$u_n = A + Bn$	A1	2.2a
$u_n = \mu(2)^n \Rightarrow \mu(2)^n = 2\mu(2)^{n-1} - \mu(2)^{n-2} + (2)^n$ $\Rightarrow 4\mu(2)^{n-2} = 4\mu(2)^{n-2} - \mu(2)^{n-2} + 4(2)^{n-2}$ $\Rightarrow \mu = \dots \{4\}$	M1	1.1b
$u_n = A + Bn + 4(2)^n$ or $u_n = A + Bn + (2)^{n+2}$	A1	1.1b
	(4)	

Notes

M1: Forms and solves the auxiliary equation.

A1: Correct complementary function.

M1: Correct form for the particular solution, substitutes into the recurrence relation to find the PS.

A1: Correct general solution

(corrected from the printed mark scheme: the first line is printed in a garbled symbol font)

Mark scheme (b)

Scheme	Marks	AO
Uses the information to find the values of the constants For example $u_0 = 2u_1 \Rightarrow A + 4 = 2(A + B + 4(2)) \Rightarrow \dots \{A + 2B = -12\}$ $u_4 = 3u_2 \Rightarrow A + 4B + '4'(2)^4 = 3(A + 2B + 4(2)^2) \Rightarrow \dots \{2A + 2B = 16\}$ Solves simultaneous equations to find values for A and B. Alternatively $u_2 = 2u_1 - u_0 + 2^2 = u_0 - u_0 + 4 = 4$ leading to	M1	3.1a

$u_4 = 3u_2 = 3 \times 4 = 12$ $u_3 = 2u_2 - u_1 + 2^3 = 2 \times 4 - \left(\frac{A+4}{2}\right) + 8 = 16 - \left(\frac{A+4}{2}\right)$ $u_4 = 2u_3 - u_2 + 2^4 = 2 \left(16 - \left(\frac{A+4}{2}\right)\right) - 4 + 16 = 12$ Leading $A = \dots$ $u_2 = '28' + 2B + 4(2^2) = 4 \Rightarrow B = \dots$		
$u_n = 28 - 20n + 4(2)^n$ or $u_n = 28 - 20n + (2)^{n+2}$	A1	1.1b
	(2)	
	(6 marks)	

Notes

Note: They must have two constants to score any marks in this part

M1: A complete method to find the constants using the information given. Form two equations and solves simultaneously to find values for A and B .

A1: Correct solution.

2.

Mark scheme (a)

Scheme	Marks	AO
Implies moves from A - Has to hop to different lily pad to A - Will not hop on the same pad - They can't stay on lily pad A - Goes to B or C but not A - Can't be where it started - Can't stay on A	B1	2.4
	(1)	

Notes

B1: Explains has to move from A

Mark scheme (b)

Scheme	Marks	AO
$p_1 = \frac{2}{3} \left(-\frac{1}{2}\right)^1 + \left(\frac{1}{3}\right) = 0$ Minimum needed $p_1 = \frac{2}{3} \left(-\frac{1}{2}\right) + \left(\frac{1}{3}\right) = 0$	B1	2.1
Assume result is true for $n = k$ Or $p_k = \frac{2}{3} \left(-\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)$	M1	2.4
Finds an expression for $p_{k+1} = \frac{1}{2} \left(1 - \left[\frac{2}{3} \left(-\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)\right]\right)$	M1	1.1b
$p_{k+1} = \frac{1}{3} + \frac{2}{3} \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right)^k$ or $\frac{1}{2} - \frac{1}{6} + \frac{2}{3} \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right)^k$ $p_{k+1} = \frac{1}{3} - \frac{1}{3} \times -2 \left(-\frac{1}{2}\right)^{k+1}$ or $\frac{1}{2} - \frac{1}{6} - \frac{1}{3} \times -2 \left(-\frac{1}{2}\right)^{k+1}$	M1	2.1
$p_{k+1} = \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} + \left(\frac{1}{3}\right)$	A1	1.1b
<u>If true for $n = k$ then it is true for $n = k + 1$</u> and as it is <u>true for $n = 1$</u> , the statement is <u>true for all n</u> . (Allow 'for all values')	A1	2.2a
	(6)	

Notes

B1: Shows the statement is true for $n = 1$. Needs to show that $p_1 = 0$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$

M1: Makes a statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then ...etc

M1: Finds an expression for p_{k+1} using the recurrence relation and substitutes in for p_k

M1: Shows a correct intermediate stage $p_{k+1} = \frac{1}{3} + \frac{2}{3} \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right)^k$

Condone $\frac{1}{3} - \frac{2}{3} \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right)^k$

A1: $p_{k+1} = \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} + \left(\frac{1}{3}\right)$ cso

A1: Correct complete conclusion. This mark is dependent on previous four marks, the B mark is not required. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Note Using $p_{k+1} = \frac{1}{2} - \frac{1}{2}p_k = \frac{1}{2} - \frac{1}{2} \left[\frac{2}{3} \left(-\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)\right] = \frac{1}{2} + \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} - \left(\frac{1}{6}\right)$ scores 2nd M1 3rd

M1

Mark scheme (c)

Scheme	Marks	AO
As $n \rightarrow \infty$, $\left(-\frac{1}{2}\right)^n \rightarrow 0 \therefore p_n \rightarrow \frac{1}{3}$	B1	3.4
	(1)	
	(8 marks)	

Notes

B1: see scheme

3.

Mark scheme (a)

Scheme	Marks	AO
$A = 50$	B1	3.3
Interest rate is 0.5% so multiplied by 1.005	B1	2.4
	(2)	

Notes

B1: Uses the model to state $A = 50$

B1: A correct explanation

Mark scheme (b)

Scheme	Marks	AO
A complete method to solve the recurrence relation using $U_n = CF + PS = c(1.005)^n + \lambda$	M1	3.1a
$PS = \lambda \Rightarrow \lambda = 1.005\lambda - \text{"50"} \text{ leading to } \lambda = \dots$	M1	1.1b
$\lambda = 10000$	A1	1.1b
Uses $U_0 = 1000$ and their value of λ to find the value of c $1000 = c(1.005)^0 + \text{"10000"}$ Leading to $c = \dots \{-9000\}$	M1	1.1b
$U_n = 10000 - 9000(1.005)^n$	A1	1.1b

Notes

M1: A complete method to solve the recurrence relation using $U_n = CF + PS = c(1.005)^n + \lambda$

M1: Uses their value of A with $PS = \lambda \Rightarrow \lambda = 1.005\lambda - "50"$ to find a value for λ

A1: $\lambda = 10000$

M1: Uses $U_0 = 1000$ and their value of λ to find the value of c

A1: Fully correctly defined sequence $U_n = 10000 - 9000(1.005)^n$

Alternative

Scheme	Marks	AO
A complete method to solve the recurrence relation using $U_n = CF + PS = c(1.005)^n + \lambda$	M1	3.1a
$U_0 = c(1.005)^0 + \lambda \Rightarrow 1000 = c + \lambda$	M1	1.1b
$U_1 = 1.005(1000) - 50 = 955$ $U_1 = c(1.005)^1 + \lambda \Rightarrow 955 = 1.005c + \lambda$	M1 A1	1.1b 1.1b
$U_n = 10000 - 9000(1.005)^n$	A1	1.1b
	(5)	

M1: A complete method to solve the recurrence relation using $U_n = CF + PS = c(1.005)^n + \lambda$

M1: Uses U_0 to form an equation for their c and their λ

M1: Uses the given formula to find U_1 and then uses their value to form another equation for their c and their λ

A1: Correct second equation using U_1

A1: Solve simultaneously to find $U_n = 10000 - 9000(1.005)^n$

Mark scheme (c)

Scheme	Marks	AO
$10000 - 9000(1.005)^n = 0 \Rightarrow n = \dots$ $(1.005)^n = \frac{10000}{9000} \Rightarrow n = \frac{\log \frac{10}{9}}{\log 1.005} = \dots$ $(1.005)^n = \frac{10000}{9000} \Rightarrow n = \log_{1.005} \frac{10}{9} = \dots$	M1	3.4

$n = 21.1$ therefore 22	A1	3.2a
	(2)	
	(9 marks)	

Notes

M1: Sets their expression = 0 and solves using logarithms to find a value for n .

A1: 22 (months)

Note: Using trail and error must show that $U_{21} > 0$ is positive and $U_{22} < 0$