

QQQ – Further Pure 2 – Chapter 4 – Recurrence Relations (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. The general solution of the first order recurrence relation

$$u_{n+1} + au_n = bn^2 + cn + d \quad n \geq 0$$

is given by

$$u_n = A(3)^n + 5n^2 + 1$$

where A is an arbitrary non-zero constant.

By considering expressions for u_{n+1} and u_n , find the values of the constants a, b, c and d . **(3)**

(Total 3 marks)

Edexcel A2 June 2024 Q2

2. Determine a closed form for the recurrence system

$$u_1 = 4 \quad u_2 = 6$$

$$u_{n+2} = 6u_{n+1} - 9u_n \quad (n = 1, 2, 3, \dots)$$

(Total 5 marks)

Edexcel A2 June 2024 Q2

3. A recurrence system is defined by

$$u_{n+2} = 9(n+1)^2 u_n - 3u_{n+1} \quad n \geq 1$$

$$u_1 = -3, \quad u_2 = 18$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = (-3)^n n!$$

(6)

(Total 6 marks)

Edexcel A2 October 2021 Q6

4. A person takes a course of a particular vitamin.

Before the course there was none of the vitamin in the person's body.

During the course, vitamin tablets are taken at the same time each day.

Initially two tablets are taken and on each following day only one tablet is taken.

Each tablet contains 10 mg of the vitamin.

Between doses the amount of the vitamin in the person's body decreases naturally by 60%

Let u_n mg be the amount of the vitamin in the person's body immediately after a tablet is taken, n days after the initial two tablets were taken.

(a) Explain why u_n satisfies the recurrence relation

$$u_0 = 20 \quad u_{n+1} = 0.4u_n + 10$$

(2)

The general solution to this recurrence relation has the form $u_n = a(0.4)^n + b$

(b) Determine the value of a and the value of b .

(4)

The course is only effective if the amount of the vitamin in the person's body remains above 6 mg at **all times** throughout the course.

(c) Determine whether this course of the vitamin will be effective for this person, giving a reason for your answer.

(3)

(Total 9 marks)

Solutions

1.

Scheme	Marks	AO
$u_{n+1} = A(3)^{n+1} + 5(n+1)^2 + 1$	B1	1.1b
$u_{n+1} = A(3)^n(3) + 5(n+1)^2 + 1$ $\Rightarrow u_{n+1} = 3(u_n - 5n^2 - 1) + 5(n+1)^2 + 1$	M1	1.1b
$u_{n+1} - 3u_n = -10n^2 + 10n + 3$	A1	1.1b
	(3)	
	(3 marks)	

Notes

B1: any correct expression for u_{n+1}

M1: eliminating A to form a first order recurrence relation containing u_{n+1} and u_n

A1: CAO for $u_{n+1} - 3u_n = -10n^2 + 10n + 3$ (need not explicitly state

$a = -3, b = -10, c = 10, d = 3$)

Alternative 1

Scheme	Marks	AO
$u_{n+1} + au_n = 0$ $u_{n+1} = -au_n$ C.F. $u_n = A(-a)^n$ $-a = 3 \Rightarrow a = -3$	B1	
Particular solution Try $\lambda n^2 + \mu n + \nu \Rightarrow \lambda = 5 \quad \mu = 0 \quad \nu = 1$ $5(n+1)^2 + 1 - 3(5n^2 + 1) = bn^2 + cn + d$ $\Rightarrow b = -10 \quad c = 10 \quad d = 3$	M1 A1	
	(3)	

B1: Considers the C.F. and deduces $a = -3$ with no other values stated

M1: Obtains P.S. and forms equation using $u_{n+1} \pm 3u_n$

A1: CAO for $u_{n+1} - 3u_n = -10n^2 + 10n + 3$ (need not explicitly state

$a = -3, b = -10, c = 10, d = 3$)

Alternative 2

Scheme	Marks	AO
$u_{n+1} + au_n = 3A(3)^n + 5n^2 + 10n + 6$ $+aA(3)^n + 5an^2 + a$	B1	
Compares coefficients $3 + a = 0 \Rightarrow a = -3$ $5 + 5a = b \Rightarrow b = -10$ $c = 10$ $6 + a = d \Rightarrow d = 3$	M1 A1	
	(3)	

B1: Forms the correct equation for $u_{n+1} + au_n$

M1: Attempts to compare coefficients – at least three terms seen

A1: CAO

2.

Scheme	Marks	AO
Aux equation is $r^2 - 6r + 9 = 0 \Rightarrow r = \dots$	M1	1.1b
$(r - 3)^2 = 0 \Rightarrow r = 3$	A1	1.1b
General form is $u_n = (A + Bn)3^n$	M1	1.1a
$\begin{cases} 4 = 3(A + B) \\ 6 = 9(A + 2B) \end{cases} \Rightarrow A = \dots, B = \dots$	M1	2.1
$u_n = \left(2 - \frac{2}{3}n\right) 3^n$ o.e $u_n = (6 - 2n)3^{n-1}$	A1	1.1b
	(5)	
	(5 marks)	

Notes

M1: Forms and solves the auxiliary equation.

A1: Correct single root of 3.

M1: Selects the correct form for u_n for their roots of the equation. (If distinct real roots were found allow for $u_n = A\alpha^n + B\beta^n$)

M1: Uses the values of u_1 and u_2 with the corresponding values of n used to form and uses a correct method to solve simultaneous equations to find the constants. If no method is shown, use of calculator, for solving the simultaneous equations the values must be correct for their equations.

A1: Correct closed form, isw

3.

Scheme	Marks	AO
$n = 1 : u_1 = (-3)^1 \times 1! = -3$ $n = 2 : u_2 = (-3)^2 \times 2! = 9 \times 2 = 18$ Hence true for $n = 1$ and $n = 2$	B1	2.2a
Assume true for some $n = k$ and $n = k + 1$, so $u_k = (-3)^k k!$ and $u_{k+1} = (-3)^{k+1} (k + 1)!$	M1	2.4
Then $u_{k+2} = 9(k + 1)^2 ((-3)^k k!) - 3((-3)^{k+1} (k + 1)!)$	M1	1.1b
$= (-3)^k k! [9(k + 1)^2 - 3(-3)(k + 1)]$	M1	1.1b
$= (-3)^k k! [9(k + 1)(k + 1 + 1)] = (-3)^k \times (-3)^2 \times (k + 1)(k + 2)k!$ $= (-3)^{k+2} (k + 2)!$	A1	2.1
Hence if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$. As also true for $n = 1$ and $n = 2$, then true for all $n \in \mathbb{N}$ by mathematical induction.	A1	2.4
	(6)	
	(6 marks)	

Notes

B1: Checks the closed form works for $n = 1$ and $n = 2$

M1: Makes the inductive assumption. May use e.g. $n = k - 2$ and $n = k - 1$ instead and show true for $n = k$. It must be clear it is the closed forms they are assuming, not a recurrence form.

M1: Substitutes expression for $n = k$ and $n = k + 1$ (or equivalents) into the recurrence formula.

M1: Takes out common factors of at least $(-3)^k k!$ in their expression, or equivalent for their assumed true values. Treatment of the (-3) must be correct, but condone invisible brackets if recovered.

Note: they may well take out more at this stage, which is fine, e.g.

$$u_{k+2} = 9(k+1)^2 ((-3)^k k!) - 3((-3)^{k+1}(k+1)!) = (-3)^{k+2}(k+1)! [(k+1) + 1]$$

A1: Simplifies correctly to the required form for their assumed true values.

A1: Correct conclusion made. Depends on all three M's and the A being gained. Must convey the ideas of 1) true for $n = 1$ and $n = 2$, 2) if true for two successive cases, it is also true for the next case and 3) a suitable conclusion that it is true for all positive n .

4.

Mark scheme (a)

Scheme	Marks	AO
- Immediately after the first tablets are taken there is 20 mg in the person, so $u_0 = 20$ - Reduction by 60% through day means just before next tablet is taken there is $0.4u_n$ mg of the vitamin left in the person - The next tablet taken adds 10 mg to the amount just before the tablet is taken, giving $u_{n+1} = 0.4u_n + 10$	B1	2.4
	B1	3.3
	(2)	

Notes

B1: For explaining any two of the three aspects in the scheme.

B1: All three aspects explained.

Mark scheme (b)

Scheme	Marks	AO
$u_1 = 0.4 \times 20 + 10 = 18$	B1	3.1a
So $\left. \begin{matrix} 20 = a + b \\ 18 = \frac{2}{5}a + b \end{matrix} \right\} \Rightarrow 20 - 18 = a \left(1 - \frac{2}{5}\right) \Rightarrow a = \dots$	M1	1.1b
$a = \frac{10}{3}$ and $b = \frac{50}{3}$	A1	1.1b
	A1	1.1b
	(4)	

Notes

B1: For $u_1 = 18$ found in order to be able to form and solve simultaneous equations.

M1: Setting up and solving equations for a and b using their u_1 and 20, with at least one value found.

A1: Either a or b correct.

A1: Both a and b correct.

Alt (b)

Scheme	Marks	AO
$n = 0 \Rightarrow 20 = a + b$	B1	3.1a
$u_{n+1} = 0.4u_n + 10 \Rightarrow a(0.4)^{n+1} + b = 0.4(a(0.4)^n + b) + 10$ $\Rightarrow b = 0.4b + 10$	M1	1.1b
$b = \frac{50}{3}$ and $a = \frac{10}{3}$	A1 A1	1.1b 1.1b
	(4)	

B1: Uses $n = 0$ to form a correct relation relating a and b .

M1: Substitutes the given general form into the recurrence relation and extracts an equation in b only.

A1: Correct value for b

A1: Both a and b correct.

Alt 2 (b)

Scheme	Marks	AO
Particular solution is $k \Rightarrow k = 0.4k + 10 \Rightarrow k = \frac{50}{3}$	B1	1.ba
$u_n = a(0.4)^n + \frac{50}{3} \Rightarrow 20 = a(0.4)^0 + \frac{50}{3}$	M1 A1ft	3.1a 1.1b
$a = \frac{10}{3}$	A1	1.1b
	(4)	

B1: Uses particular solution is constant to find b

M1: Substitutes for $n = 0$ or $n = 1$ and sets equal to 20 in the general form to form an equation in a

A1: Correct equation follow through their b .

A1: a correct.

Mark scheme (c)

Scheme	Marks	AO
In long term $u_n = \frac{10}{3}(0.4)^n + \frac{50}{3} \rightarrow \frac{50}{3}$ as $(0.4)^n \rightarrow 0$	M1	3.4
Minimum amount of vitamin occurs just before a tablet is taken, so is $\frac{50}{3} - 10 = \frac{20}{3} = 6\frac{2}{3}$ mg	M1	3.1b
This is greater than 6 mg and so there is always at least 6 mg of the vitamin in the person. The course of vitamin will be effective.	A1	3.2a
	(3)	
	(9 marks)	

Notes

M1: Uses the model to work out the long-term behaviour or the amount of vitamin in the person. Accept identifying $\frac{50}{3}$ as a lower limit for u_n .

M1: Finds the lower bound for the amount of vitamin in the person, subtracting 10 from their long-term value or multiplying the $\frac{50}{3}$ limit by 0.4 to find amount just before next tablet.

A1: Concludes the course of vitamin will be effective following correct work and reasoning.

NB: Attempts that set $u_n = 6$ and try to solve for n score no marks. Attempts that set $u_n - 10 = 6$ and try to solve for n can score the second M only. Long term behaviour needs to be considered for first M.