

QQQ – Further Pure 2 – Chapter 3 – Complex Numbers (Version 2)

Total Marks: 24

(24 = Platinum, 22 = Gold, 19 = Silver, 16 = Bronze)

1.

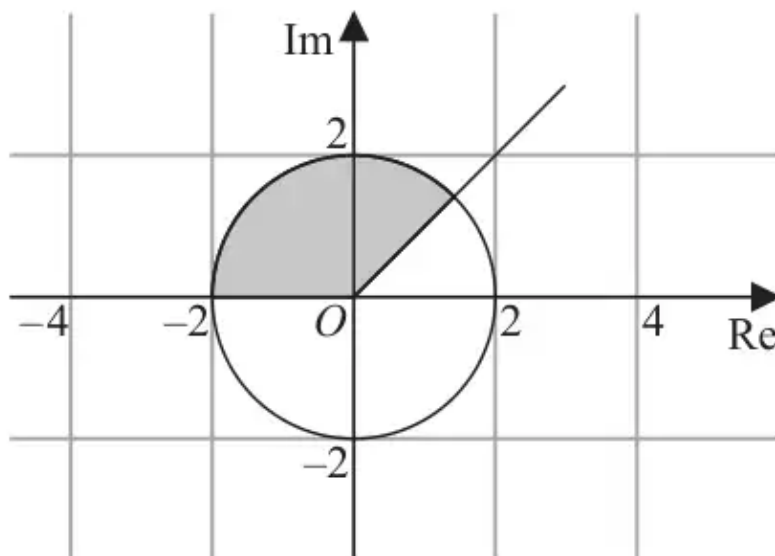


Figure 1

Figure 1 shows an Argand diagram.

The set of points, A , that lies within the shaded region, including its boundaries, is defined by

$$A = \{z : p \leq \arg(z) \leq q\} \cap \{z : |z| \leq r\}$$

where p , q and r are positive constants.

(a) Write down the values of p , q and r . (2)

Given that $w = -2\sqrt{3} + 2i$ and $z \in A$,

(b) find the maximum value of $|w - z|^2$ giving your answer in an exact simplified form. (4)

(Total 6 marks)

2. A transformation T from the z -plane to the w -plane is given by

$$w = \frac{z + 2i}{iz} \quad z \neq 0$$

The transformation maps points on the real axis in the z -plane onto a line in the w -plane.

Find an equation of this line.

(4)

(Total 4 marks)

Edexcel FP2 June 2013 (R) Q1

3.

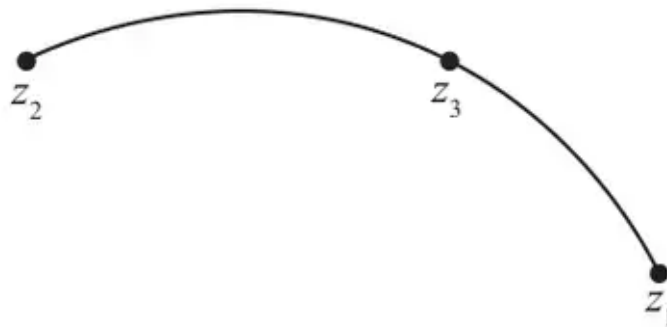


Figure 1

Figure 1 shows a locus in the complex plane.

The locus is an arc of a circle from the point represented by $z_1 = 3 + 2i$ to the point represented by $z_2 = a + 4i$, where a is a constant, $a \neq 1$

Given that

- the point $z_3 = 1 + 4i$ also lies on the locus
- the centre of the circle has real part equal to -1

(a) determine the value of a . (2)

(b) Hence determine a complex equation for the locus, giving any angles in the equation as positive values. (3)

(Total 5 marks)

Edexcel A2 June 2023 Q9

4.

- (i) A circle C in the complex plane is defined by the locus of points satisfying

$$|z - 3i| = 2|z|$$

- (a) Determine a Cartesian equation for C , giving your answer in simplest form. (3)

- (b) On an Argand diagram, shade the region defined by

$$\{z \in \mathbb{C} : |z - 3i| > 2|z|\}$$

(2)

- (ii) The transformation T from the z -plane to the w -plane is given by

$$w = z^3$$

- (a) Describe the geometric effect of T . (2)

The region R in the z -plane is given by

$$\left\{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{4}\right\}$$

- (b) On a **different** Argand diagram, sketch the image of R under T . (2)

(Total 9 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$p = \frac{\pi}{4}$ and $q = \pi$ or $\frac{\pi}{4} \leq \arg z \leq \pi$	B1	1.1b
$r = 2$ or $ z \leq 2$	B1	1.1b
	(2)	

Notes

B1: Correct values or correct loci

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Mark scheme (b)

Scheme	Marks	AO
Position for z is at intersection in quadrant 1	B1	3.1a
Angle between $y = x$ and “ OW ” is $\frac{\pi}{3} + \frac{\pi}{4}$	B1	2.2a
$d^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$	M1	1.1b
$= 20 - 4\sqrt{2} + 4\sqrt{6}$	A1	2.2a
	(4)	
	(6 marks)	

Notes

B1: Realises that the position for z is at the intersection in quadrant 1 and makes progress in finding the distance

B1: Deduces from the information given that the angle between $y = x$ and “ OW ” is $\pi/4 + \pi/3$

M1: Correct use of the cosine rule in order to find the required length²

A1: Deduces the required value in a simplified and exact form

Alternative for (b)

Scheme	Marks	AO
Position for z is at intersection in quadrant 1	B1	3.1a

$y = x$ intersects $x^2 + y^2 = 4$ at $(\sqrt{2}, \sqrt{2})$	B1	2.2a
$d^2 = (\sqrt{2} + 2\sqrt{3})^2 + (\sqrt{2} - 2)^2$	M1	1.1b
$= 20 - 4\sqrt{2} + 4\sqrt{6}$	A1	2.2a
	(4)	

B1: Realises that the position for z is at the intersection in quadrant 1 and makes progress in finding the distance

B1: Deduces from the information given the correct coordinates for z

M1: Correct use of Pythagoras in order to find the required length²

A1: Deduces the required value in a simplified and exact form

2.

Scheme	Marks
$z = x \quad w = \frac{x + 2i}{ix}$	M1A1
$w = \frac{1}{i} + \frac{2i}{ix}$	
$u + iv = -i + \frac{2}{x}$	
$\left(u = \frac{2}{x}\right) \quad v = -1$	M1
$\therefore w$ is on the line $v + 1 = 0$	A1
	(4 marks)

Notes

M1 for replacing at least one z with x to obtain (ie show an appreciation that $y = 0$)

A1 $w = \frac{x + 2i}{ix}$

M1 for writing w as $u + iv$ and equating real or imaginary parts to obtain either u or v in terms of x or just a number

A1 for giving the equation of the line $v + 1 = 0$ or must be in terms of v

Q1 - Alternative 1

Scheme	Marks
$w = \frac{x + iy + 2i}{i(x + iy)}$ Replacing z with $x + iy$	

$w = \frac{x + iy + 2i}{-y + ix} \times \frac{-y - ix}{-y - ix}$	
$w = \frac{(x + i(y + 2))(-y - ix)}{y^2 + x^2}$	
$w = \frac{2x - i(x^2 + y^2 + 2y)}{y^2 + x^2}$	
$w = \frac{2x - ix^2}{x^2} = \frac{2}{x} - i$ Using $y = 0$. This is where the first M1 may be awarded. A1 if correct even if expression is unsimplified but denominator must be real	M1A1
$v = -1$ M1, A1 as in main scheme	M1A1

Q1 – Alternative 2

Scheme	Marks
$z = \frac{2i}{iw - 1}$ Writing the transformation as a function of w	
$z = \frac{2i}{i(u + iv) - 1}$	
$z = \frac{2i}{(-v - 1) + iu} \times \frac{(-v - 1) - iu}{(-v - 1) - iu}$	
$z = \frac{2u + 2i(-v - 1)}{(-v - 1)^2 + u^2} = \frac{2u}{(-v - 1)^2 + u^2} + i \left(\frac{2(-v - 1)}{(-v - 1)^2 + u^2} \right)$	
$\left(\frac{2(-v - 1)}{(-v - 1)^2 + u^2} \right) = 0$ or simply $-2(v + 1) = 0$ Using $y = 0$. This is where the first M1 may be awarded. A1 if correct even if expression is unsimplified but denominator must be real	M1A1
$v = -1$ M1, A1 as in main mark scheme above	M1A1

3.

Mark scheme (a)

Scheme	Marks	AO
Centre lies on perpendicular bisector of z_2 and z_3, which is vertical as same imaginary components. Hence $\frac{a+1}{2} = -1 \Rightarrow a = \dots$ Or $(a - (-1))^2 + (4 - b)^2 = (1 - (-1))^2 + (4 - b)^2 \Rightarrow a = \dots$ $(a - (-1))^2 = (1 - (-1))^2 \Rightarrow a = \dots$ Or $-1 - a = 1 - (-1) \Rightarrow a = \dots$	M1	1.1b

$a = -3$	A1	2.2a
	(2)	

Notes

M: Forms a correct strategy to find the value of a , see scheme for various approaches

A1: Correct value

Alternative

Scheme	Marks	AO
$\left. \begin{aligned} (x - (-1))^2 + (y - b)^2 &= r^2 \\ (1 + 1)^2 + (4 - b)^2 &= r^2 \\ (3 + 1)^2 + (2 - b)^2 &= r^2 \end{aligned} \right\} \Rightarrow 4 + (4 - b)^2 = 16 + (2 - b)^2 \Rightarrow b = \dots \{0\}$ $r^2 = 4 + (4 - "0")^2 = \dots \{20\}$ $(a + 1)^2 + (4 - "0")^2 = "20" \Rightarrow a = \dots$	M1	1.1b
$a = -3$ only the other root must be rejected	A1	2.2a
	(2)	

Mark scheme (b)

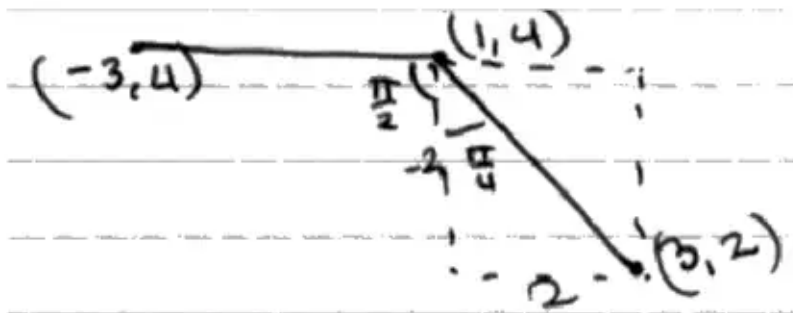
Scheme	Marks	AO
Equation has form $\arg \left(\frac{z - z_1}{z - z_2} \right) = \theta$	B1	1.2
$\arg \left(\frac{1 + 4i - (3 + 2i)}{1 + 4i - ("3" + 4i)} \right) = \arg(-2 + 2i) - \arg(4) = \dots \left(= \frac{3\pi}{4} \right)$ Or $\arg \left(\frac{1 + 4i - 3 - 2i}{1 + 4i - a - 4i} \right) = \arg \left(\frac{-2 + 2i}{4} \right) = \arg \left(-\frac{1}{2} + \frac{1}{2}i \right) = \frac{3\pi}{4}$ or $\overrightarrow{z_3 z_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{z_3 z_1} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ $\cos \theta = \frac{\begin{pmatrix} -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix}}{4\sqrt{2^2 + (-2)^2}} \Rightarrow \theta = \dots$	M1	3.1a

Or

$$\cos \theta = \frac{4^2 + 8 - 40}{2 \times 4 \times \sqrt{8}} \Rightarrow \theta = \dots$$

or

Using trigonometry



Equation is $\arg\left(\frac{z-3-2i}{z+3-4i}\right) = \frac{3\pi}{4}$ o.e.e.

Equation is $\arg(z - 3 - 2i) - \arg(z + 3 - 4i) = \frac{3\pi}{4}$ o.e.e.

A1

2.1

(3)

(5 marks)

Notes

B1: Recalls the correct form for the equation of an arc, with any angle or θ . Look for the correct form, so allow if z_1 and z_2 are the other way round.

M1: Any correct full method to find the value of θ , see scheme for various approaches

A1: Correct equation.

Correct answer implies full marks

4.

Mark scheme (i)(a)

Scheme	Marks	AO
$ x + (y - 3)i = 2 x + yi \Rightarrow x^2 + (y - 3)^2 = 4(x^2 + y^2)$	M1	3.1a
$\Rightarrow x^2 + y^2 - 6y + 9 = 4x^2 + 4y^2 \Rightarrow x^2 + y^2 + 2y - 3 = 0$ in any order	M1 A1	1.1b 1.1b
	(3)	

Notes

M1: Substitutes $z = x + yi$ into the equation and applies the modulus to obtain an equation with no i's. Must have dealt with the i^2 correctly. Condone not squaring the 2.

M1: Expands and gathers terms. Must have an x^2 and y^2 after simplifying so that it is an equation of a circle.

A1: Cancels common factor 3 to obtain the equation shown. ISW if they make errors trying to complete the square.

Note: Any letters may be used not just x and y

Alt (a)

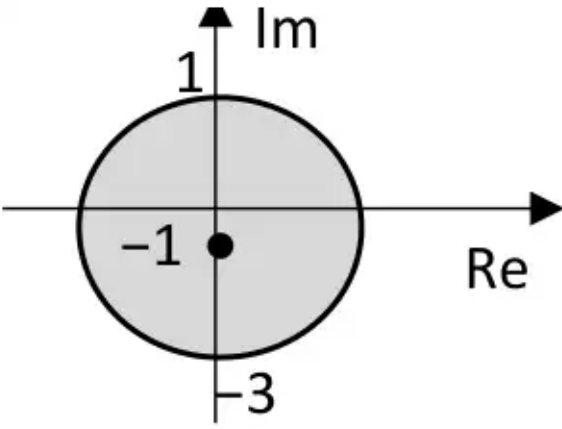
Scheme	Marks	AO
Points are twice as far from $3i$ as from 0 so i and $-3i$ are diametrically opposite points.	M1	3.1a
So radius is $\frac{ i - (-3i) }{2} = 2$ and centre is $\frac{i - 3i}{2} = -i$	M1	1.1b
Hence equation is $x^2 + (y + 1)^2 = 4$	A1	1.1b
	(3)	

M1: Identifies two diametrically opposite points on the circle by understanding the geometry of the situation.

M1: Finds the radius and centre of the circle from their points, dotted or solid line

A1: Correct equation need not expand, but should be in simplest form.

Mark scheme (i)(b)

Scheme	Marks	AO
		
Circle drawn with the inside shaded	M1	2.2a
Correct circle drawn for their equation. Implied by the position of the centre and radius. Inside shaded.	A1ft	3.1a
	(2)	

Notes

M1: Circle drawn anywhere and the inside shaded

A1ft: Correct circle drawn for their equation. Implied by the position of the centre and radius. Inside shaded.

Mark scheme (ii)(a)

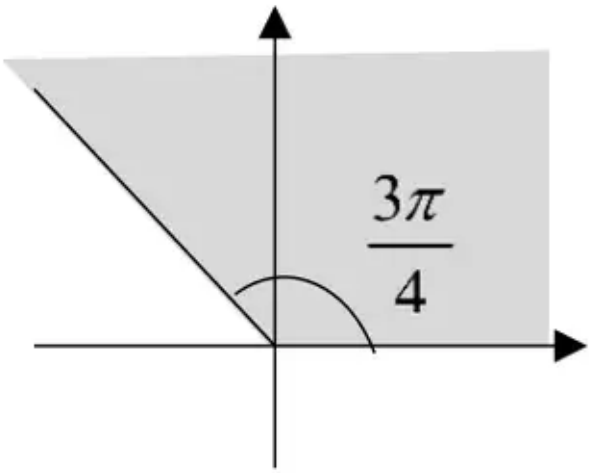
Scheme	Marks	AO
A point z is mapped to a point with 3 times the argument... Rotate every point by 2 times the argument	B1	2.4
... and with modulus as the modulus of the cube of z .	B1	2.5
	(2)	

Notes

B1: Identifies that the argument triples in size.

B1: Identifies that the modulus scales according to the modulus of z cubed.

Mark scheme (ii)(b)

Scheme	Marks	AO
		
Sector from O along the real axis, indicated by some shading, in an anticlockwise direction	M1	1.1b
Correct sector, angle must be stated or implied by the diagram	A1	2.2a
	(2)	
	(9 marks)	

Notes

M1: A sector centre O and starting along the real axis and in an anticlockwise direction. Must be some shading to represent the region, dotted or solid line

A1: Correct sector shaded and the angle stated or implied