

QQQ – Further Pure 2 – Chapter 2 – Groups (Version 2)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. The set $G = \{1, 3, 7, 9, 11, 13, 17, 19\}$ under the binary operation of multiplication modulo 20 forms a group.

- (a) Find the inverse of each element of G . (3)
- (b) Find the order of each element of G . (3)
- (c) Find a subgroup of G of order 4 (1)
- (d) Explain how the subgroup you found in part (c) satisfies Lagrange's theorem. (1)

(Total 8 marks)

Edexcel AS October 2020 Q1

2. A binary operation $\star$ on the set of non-negative integers, $\mathbb{Z}_0^+$, is defined by

$$m \star n = |m - n| \quad m, n \in \mathbb{Z}_0^+$$

- (a) Explain why $\mathbb{Z}_0^+$ is closed under the operation $\star$ (1)
- (b) Show that 0 is an identity for $(\mathbb{Z}_0^+, \star)$ (2)
- (c) Show that all elements of $\mathbb{Z}_0^+$ have an inverse under $\star$ (2)
- (d) Determine if $\mathbb{Z}_0^+$ forms a group under $\star$, giving clear justification for your answer. (3)

(Total 8 marks)

Edexcel A2 October 2021 Q2

3. The set $S = \{1, 3, 5, 9, 11, 13\}$ forms the group G , under the operation multiplication modulo 14

(a) Complete the Cayley table below for the group G

$\times_{14}$	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11			
9	9	13		11		
11	11	5			9	
13	13	11				1

(3)

(b) Write down a subgroup of G of order 2

(1)

The group H is defined by the Cayley table below.

$*$	p	q	r	s	t	u
p	p	q	r	s	t	u
q	q	t	u	r	s	p
r	r	u	t	q	p	s
s	s	r	q	p	u	t
t	t	s	p	u	r	q
u	u	p	s	t	q	r

(c) Show that G and H are isomorphic.

(3)

(Total 7 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO								
1, 9, 11 and 19 are self-inverse	M1 A1	1.1b 1.1b								
<table><tr><td>3</td><td>7</td><td>13</td><td>17</td></tr><tr><td>7</td><td>3</td><td>17</td><td>13</td></tr></table>	3	7	13	17	7	3	17	13	B1	1.1b
3	7	13	17							
7	3	17	13							
	(3)									

Notes

M1: For any 2 of the self-inverse elements

A1: All 4 self-inverse elements correctly identified

B1: Correct inverses for the other elements

Mark scheme (b)

Scheme	Marks	AO																
<table><tr><td>1</td><td>3</td><td>7</td><td>9</td><td>11</td><td>13</td><td>17</td><td>19</td></tr><tr><td>1</td><td>4</td><td>4</td><td>2</td><td>2</td><td>4</td><td>4</td><td>2</td></tr></table>	1	3	7	9	11	13	17	19	1	4	4	2	2	4	4	2	M1 A1 A1	1.1b 1.1b 1.1b
1	3	7	9	11	13	17	19											
1	4	4	2	2	4	4	2											
	(3)																	

Notes

M1: At least 3 correct orders

A1: 6 correct orders

A1: All correct

Mark scheme (c)

Scheme	Marks	AO
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$\{1, 3, 7, 9\}$ or $\{1, 9, 13, 17\}$ or $\{1, 9, 11, 19\}$	B1	2.5
	(1)	

Notes

B1: Describes a correct subgroup of order 4

Mark scheme (d)

Scheme	Marks	AO
Because 4 is a factor of 8	B1	2.4
	(1)	
	(8 marks)	

Notes

B1: Correct explanation

2.

Mark scheme (a)

Scheme	Marks	AO
For $m, n \in \mathbb{Z}_0^+$ we have $m - n \in \mathbb{Z}$ (difference of integers is an integer) and so $ m - n \in \mathbb{Z}_0^+$, hence closed under $\star$.	B1	2.4
	(1)	

Notes

B1: Checks difference of two non-negative integers is an integer and hence its modulus is a non-negative integer and concludes closure. "Always positive" as a conclusion is B0 without consideration of the equal zero case.

Mark scheme (b)

Scheme	Marks	AO
For $m \in \mathbb{Z}_0^+$, $0 \star m = 0 - m = -m = m$	M1	1.1b
and $m \star 0 = m - 0 = m = m$ Hence 0 is an identity*.	A1*	2.1

	(2)	
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Notes

M1: Checks that 0 is a left or a right identity. (Checks either side)

A1*: Checks 0 works both sides as an identity and makes conclusion it is an identity. (Checks both sides and makes conclusion.)

Mark scheme (c)

Scheme	Marks	AO
For $m \in \mathbb{Z}_0^+$, we need $ m - n = 0 \Rightarrow n = \dots$ or shows $ m - m = 0 = 0$	M1	2.2a
As $ m - m = 0$ for all $m \in \mathbb{Z}_0^+$ each m is self-inverse.	A1	2.1
	(2)	

Notes

M1: Realises m must be its own inverse for each m – accept if just stated m is self-inverse with no proof, or if an attempt is made to show it is self-inverse, or for an attempt to solve $|m - n| = 0$

A1: Each element is self-inverse with a **full** proof given.

Mark scheme (d)

Scheme	Marks	AO
Checks associativity – ie evaluates $m \star (n \star p)$ and $(m \star n) \star p$ with letter or numbers.	M1	1.2
E.g. $1 \star (2 \star 3) = 1 \star 2 - 3 = 1 \star 1 = 0$ but $(1 \star 2) \star 3 = 1 - 2 \star 3 = 1 \star 3 = 1 - 3 = 2$	M1	3.1a
$1 \star (2 \star 3) \neq (1 \star 2) \star 3$ hence not associative so not a group.	A1	2.4
	(3)	
	(8 marks)	

Notes

M1: Realises associativity must be checked in some way – may be by producing a counter example, or by attempting to evaluate both sides of the associativity axiom for a general case. A statement of the correct identity is sufficient for the mark to be awarded.

M1: Produces a suitable counter example and evaluates both sides of associativity equation. Attempts at algebraic proofs are unlikely to succeed but allow the method for e.g consideration of $m > n > p$ giving $||m - n| - p| = |m - n - p|$ and $|m - |n - p|| = |m - n + p|$ but must have a correct reason to disambiguate the inner moduli. If in doubt use review.

A1: Must have provided a counter example. Deduces associativity does not hold and concludes $\mathbb{Z}_0^+$ is not a group under $\star$

3.

Mark scheme (a)

Scheme	Marks	AO																																																	
<table><tr><td>$\times_{14}$</td><td>1</td><td>3</td><td>5</td><td>9</td><td>11</td><td>13</td></tr><tr><td>1</td><td>1</td><td>3</td><td>5</td><td>9</td><td>11</td><td>13</td></tr><tr><td>3</td><td>3</td><td>9</td><td>1</td><td>13</td><td>5</td><td>11</td></tr><tr><td>5</td><td>5</td><td>1</td><td>11</td><td>3</td><td>13</td><td>9</td></tr><tr><td>9</td><td>9</td><td>13</td><td>3</td><td>11</td><td>1</td><td>5</td></tr><tr><td>11</td><td>11</td><td>5</td><td>13</td><td>1</td><td>9</td><td>3</td></tr><tr><td>13</td><td>13</td><td>11</td><td>9</td><td>5</td><td>3</td><td>1</td></tr></table>	$\times_{14}$	1	3	5	9	11	13	1	1	3	5	9	11	13	3	3	9	1	13	5	11	5	5	1	11	3	13	9	9	9	13	3	11	1	5	11	11	5	13	1	9	3	13	13	11	9	5	3	1	M1 A1 A1	1.1b 1.1b 1.1b
$\times_{14}$	1	3	5	9	11	13																																													
1	1	3	5	9	11	13																																													
3	3	9	1	13	5	11																																													
5	5	1	11	3	13	9																																													
9	9	13	3	11	1	5																																													
11	11	5	13	1	9	3																																													
13	13	11	9	5	3	1																																													
	(3)																																																		

Notes

M1: At least 3 correct entries.

A1: At least 6 correct entries.

A1: All 12 entries correct.

NB for the final A all entries must be given modulo 14. Do not accept e.g. 17 for 3.

Mark scheme (b)

Scheme	Marks	AO
$\{1, 13\}$	B1	2.2a
	(1)	

Notes

B1: Deduces the correct subgroup. This is the only subgroup of order 2. Accept with set brackets or round brackets as long as both elements are listed. Accept $\langle 13 \rangle$ as long as the angled brackets are clear.

Mark scheme (c)

Scheme	Marks	AO
States the groups are cyclic or identifies a generator for each. E.g. $3^2 = 9, 3^3 = 13, 3^4 = 11, 3^5 = 5, 3^6 = 1$ so 3 generates G or $q^2 = t, q^3 = s, q^4 = r, q^5 = u, q^6 = p$ so q generates H	M1	3.1a
E.g. ($3^2 = 9, 3^3 = 13, 3^4 = 11, 3^5 = 5, 3^6 = 1$) 3 (or 5) generates G and ($q^2 = t, q^3 = s, q^4 = r, q^5 = u, q^6 = p$) q (or u) generates H	A1	1.1b
Both groups are cyclic of order 6 and so are isomorphic.	A1	2.1
	(3)	
	(7 marks)	

Notes

M1: State the groups are cyclic, or identifies a generator for each group.

A1: Correctly generator identified for each group.

A1: Correct work for showing a generator for each group and conclude both cyclic and hence isomorphic.

Alt I

Scheme	Marks	AO																					
<table><tr><td>G</td><td>1</td><td>3</td><td>5</td><td>9</td><td>11</td><td>13</td></tr><tr><td>H</td><td>p</td><td>q</td><td>u</td><td>t</td><td>r</td><td>s</td></tr><tr><td>Or H</td><td>p</td><td>u</td><td>q</td><td>r</td><td>t</td><td>s</td></tr></table>	G	1	3	5	9	11	13	H	p	q	u	t	r	s	Or H	p	u	q	r	t	s	M1 A1	3.1a 1.1b
G	1	3	5	9	11	13																	
H	p	q	u	t	r	s																	
Or H	p	u	q	r	t	s																	
Also $3^2 = 9$ and $\theta(3)^2 = q^2 = t = \theta(9) = \theta(3^2)$ so structure is preserved and therefore groups are isomorphic	A1	2.1																					
	(3)																						

M1: Attempts to identify an isomorphism between the elements – may be implied by

- identifying at least 4 correct pairings
- by attempting to rearrange group tables to have the same structure

A1: For a correct pairing of all items for an isomorphism. Note that $3^2 = 9$ while $q^2 = t$ and this can be used to see if they have a fully correct pairing (check if $3 \leftrightarrow q$ then you need $9 \leftrightarrow t$, or if $3 \leftrightarrow u$ then you need $9 \leftrightarrow r$)

A1: All correct pairings with demonstration of structure being preserved for at least one of the possible choices made, or via matching of group tables to show same structure, and concludes isomorphic.

NB there are two choices for isomorphism, $1 \leftrightarrow p$ and $13 \leftrightarrow s$ are forced, 3 and 5 must map to q and u either way round, but then 9 and 11 need to be correctly selected for the order of 3 and 5 – and this must be shown for fully marks. They may likely choose the pairings by considering orders (as in Alt II) but still need a reason for the final choice.

Alt II

Scheme	Marks	AO																												
<table><tr><td>G</td><td>1</td><td>3</td><td>5</td><td>9</td><td>11</td><td>13</td></tr><tr><td>Order</td><td>identity</td><td>6</td><td>6</td><td>3</td><td>3</td><td>2</td></tr></table> <table><tr><td>H</td><td>p</td><td>q</td><td>r</td><td>s</td><td>t</td><td>u</td></tr><tr><td>Order</td><td>identity</td><td>6</td><td>3</td><td>2</td><td>3</td><td>6</td></tr></table>	G	1	3	5	9	11	13	Order	identity	6	6	3	3	2	H	p	q	r	s	t	u	Order	identity	6	3	2	3	6	M1 A1	3.1a 1.1b
G	1	3	5	9	11	13																								
Order	identity	6	6	3	3	2																								
H	p	q	r	s	t	u																								
Order	identity	6	3	2	3	6																								
Since orders match and group is order 6 (so one of only two types) therefore the groups are isomorphic isomorphic	A1	2.1																												
	(3)																													

M1: Attempts to find the order for each element of a both groups – at least four correct for at least one of the groups.

A1: All orders correct for all elements of both groups.

A1: All orders correct, reason referring to small group order and therefore groups are **isomorphic**. Note if they go on to give an incorrect pairing it is wrong reasoning so A0.

Note: In general matching orders of all elements is a necessary but not sufficient requirement for an isomorphism (e.g. for groups of order 16 there are distinct isomorphism classes which have the same number of elements of the same orders). However, for small groups (up to order 15) with few isomorphism classes this criterion is sufficient but for a fully correct answer some reference to the group order must be given.