

QQQ – Further Pure 2 – Chapter 2 – Groups (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. The set $G = \mathbb{R} - \left\{ -\frac{3}{2} \right\}$ with the operation of $x \bullet y = 3(x + y + 1) + 2xy$ forms a group.
- (a) Determine the identity element of this group. (2)
- (b) Determine the inverse of a general element x in this group. (3)
- (c) Explain why the value $-\frac{3}{2}$ must be excluded from G in order for this to be a group. (1)

(Total 6 marks)

2. The set $\{e, p, q, r, s\}$ forms a group, A , under the operation $*$

Given that e is the identity element and that

$$p * p = s \quad s * s = r \quad p * p * p = q$$

(a) show that

(i) $p * q = r$

(ii) $s * p = q$

(2)

(b) Hence complete the Cayley table below.

$*$	e	p	q	r	s
e					
p					
q					
r					
s					

(2)

(c) Use your table to find $p * q * r * s$

(1)

A student states that there is a subgroup of A of order 3

(d) Comment on the validity of this statement, giving a reason for your answer.

(2)

(Total 7 marks)

3. The set of matrices $G = \{\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with the operation $\otimes_2$ of matrix multiplication with entries evaluated modulo 2, forms a group.

(a) Show that $\mathbf{B}$ is an element of order 3 in G . **(2)**

(b) Determine the orders of the other elements of G . **(3)**

(c) Give a reason why G is **not** isomorphic to

(i) a cyclic group of order 6

(ii) the group of symmetries of a regular hexagon. **(2)**

The group H of permutations of the numbers 1, 2 and 3 contains the following elements, denoted in two-line notation,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(d) Determine an isomorphism between the groups G and H . **(3)**

(Total 10 marks)