

QQQ – Further Pure 2 – Chapter 2 – Groups (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. The set $G = \mathbb{R} - \left\{ -\frac{3}{2} \right\}$ with the operation of $x \bullet y = 3(x + y + 1) + 2xy$ forms a group.
- (a) Determine the identity element of this group. (2)
- (b) Determine the inverse of a general element x in this group. (3)
- (c) Explain why the value $-\frac{3}{2}$ must be excluded from G in order for this to be a group. (1)

(Total 6 marks)

2. The set $\{e, p, q, r, s\}$ forms a group, A , under the operation $*$

Given that e is the identity element and that

$$p * p = s \qquad s * s = r \qquad p * p * p = q$$

(a) show that

(i) $p * q = r$

(ii) $s * p = q$

(2)

(b) Hence complete the Cayley table below.

$*$	e	p	q	r	s
e					
p					
q					
r					
s					

(2)

(c) Use your table to find $p * q * r * s$

(1)

A student states that there is a subgroup of A of order 3

(d) Comment on the validity of this statement, giving a reason for your answer.

(2)

(Total 7 marks)

3. The set of matrices $G = \{\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with the operation $\otimes_2$ of matrix multiplication with entries evaluated modulo 2, forms a group.

(a) Show that $\mathbf{B}$ is an element of order 3 in G . **(2)**

(b) Determine the orders of the other elements of G . **(3)**

(c) Give a reason why G is **not** isomorphic to

(i) a cyclic group of order 6

(ii) the group of symmetries of a regular hexagon. **(2)**

The group H of permutations of the numbers 1, 2 and 3 contains the following elements, denoted in two-line notation,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(d) Determine an isomorphism between the groups G and H . **(3)**

(Total 10 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$x \bullet e = x \Rightarrow 3(x + e + 1) + 2xe = x$ $\Rightarrow 3e + 2xe = x - 3x - 3 \Rightarrow e = \dots$ Or Let $e = -1 \Rightarrow e \bullet y = 3(-1 + y + 1) + 2(-1)y = \dots$	M1	2.1
$\Rightarrow e = -\frac{2x+3}{2x+3} \Rightarrow e = -1$ Or $e \bullet y = 3(-1 + y + 1) + 2(-1)y = y$ therefore $e = -1$	A1	2.2a
	(2)	

Notes

M1: Sets up and solves a correct equation for the identity element.

Alternatively spots the identity is -1 and attempts to show this satisfies the identity property.

E.g. $-1 \bullet x = 3(-1 + x + 1) + 2(-1)x = \dots$

A1: Correct identity / correct demonstration that -1 is the identity with appropriate conclusion. (Note only one side needs to be checked as we are told G is a group so one side is sufficient by uniqueness).

Special case: states $e = -1$ with no working scores M1A0

Mark scheme (b)

Scheme	Marks	AO
Let the inverse of x be y then $x \bullet y = e \Rightarrow 3(x + y + 1) + 2xy = -1$	M1	3.1a
$\Rightarrow 3y + 2xy = -1 - 3x - 3 \Rightarrow y = \dots$	M1	2.1
$\Rightarrow x^{-1} = -\frac{3x+4}{2x+3} \text{ oe such as } -\frac{3}{2} + \frac{1}{4x+6}$	A1	2.2a

	(3)	
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Notes

M1: Sets up a correct equation for the inverse using their identity element. May use x^{-1} throughout, or may use y or other variable.

M1: Expands, rearranges and by factorising out the inverse element from two terms to find an expression for the inverse element.

A1: Correct inverse element. Accept equivalent forms.

Mark scheme (c)

Scheme	Marks	AO
E.g. for $x = -\frac{3}{2}$ the identity property would fail since we cannot divide by $2x + 3 = 0$ or shows $x^{-1} = -\frac{-\frac{1}{2}}{0}$ so no inverse or can't divide by 0 Or $-\frac{3}{2} \bullet x = 3(-\frac{3}{2} + x + 1) - 3x = -\frac{3}{2}$ so the identity property fails as not unique as any x would be an identity for $-\frac{3}{2}$.	B1	2.4
	(1)	
	(6 marks)	

Notes

B1: Explains that the identity property would fail for $x = -\frac{3}{2}$. Alternatively accept a similar argument that $x = -\frac{3}{2}$ could not have an inverse due to dividing by zero. May refer back to work in (b), if done there, without need to repeat it.

Just saying undefined is B0 without a reason why.

Note: Reasons saying closure or associativity would fail are incorrect so B0, as both of these properties are satisfied on all of $\mathbb{R}$

2.

Mark scheme (a)

Scheme	Marks	AO
$p * q = p * p * p * p = s * s = r$ OR	B1	2.1

$s * s = r \Rightarrow p * p * p * p = r \Rightarrow p * q = r$		
$s * p = p * p * p = q$ OR as $p * p * p = q$ and $p * p = s \Rightarrow s * p = q$	B1	2.1
	(2)	

Notes

B1: Correct proof to achieve the printed statement

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Mark scheme (b)

Scheme	Marks	AO																																				
<table><tr><td>*</td><td><i>e</i></td><td><i>p</i></td><td><i>q</i></td><td><i>r</i></td><td><i>s</i></td></tr><tr><td><i>e</i></td><td><i>e</i></td><td><i>p</i></td><td><i>q</i></td><td><i>r</i></td><td><i>s</i></td></tr><tr><td><i>p</i></td><td><i>p</i></td><td><i>s</i></td><td><i>r</i></td><td><i>e</i></td><td><i>q</i></td></tr><tr><td><i>q</i></td><td><i>q</i></td><td><i>r</i></td><td><i>p</i></td><td><i>s</i></td><td><i>e</i></td></tr><tr><td><i>r</i></td><td><i>r</i></td><td><i>e</i></td><td><i>s</i></td><td><i>q</i></td><td><i>p</i></td></tr><tr><td><i>s</i></td><td><i>s</i></td><td><i>q</i></td><td><i>e</i></td><td><i>p</i></td><td><i>r</i></td></tr></table>	*	<i>e</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>	<i>e</i>	<i>e</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>	<i>p</i>	<i>p</i>	<i>s</i>	<i>r</i>	<i>e</i>	<i>q</i>	<i>q</i>	<i>q</i>	<i>r</i>	<i>p</i>	<i>s</i>	<i>e</i>	<i>r</i>	<i>r</i>	<i>e</i>	<i>s</i>	<i>q</i>	<i>p</i>	<i>s</i>	<i>s</i>	<i>q</i>	<i>e</i>	<i>p</i>	<i>r</i>	M1 A1	1.1b 1.1b
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	(2)																																					

Notes

(b) Marked B1 B1 on ePen

M1: Finds at least 13 correct entries – usually the highlighted

A1: Completely correct table

(The entries highlighted in the printed mark scheme are shown in bold.)

Mark scheme (c)

Scheme	Marks	AO
$p * q * r * s = e$	B1	1.1b

	(1)	
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Notes

B1: See scheme

Mark scheme (d)

Scheme	Marks	AO
The order of a subgroup is a factor of the order of the group (Lagrange's Theorem)	M1	1.2
As 3 is not a factor of 5, the student's statement is wrong	A1	2.3
	(2)	
	(7 marks)	

Notes

M1: Some indication that the order of a subgroup must be a factor of the order of the group. May say that 3 is not a factor of 5 or equivalent

A1: Fully correct unambiguous statement that refers Lagrange's theorem and either

- 3 is not a factor of 5
- 3 does not divide 5
- 5 is not divisible by 3

and comments that the student's statement is incorrect. No contradictory statements

3.

Mark scheme (a)

Scheme	Marks	AO
$\mathbf{B}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \{= \mathbf{E}\}$ Accept $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as long as $2 \equiv 0 \pmod{2}$ is used later. Or $\mathbf{B}^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{= \mathbf{I}\}$	M1	2.5

Accept $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ as long as $2 \equiv 0 \pmod{2}$ and $3 \equiv 1 \pmod{2}$ is used later.		
$\mathbf{B}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{= \mathbf{I}\}$ (and $\mathbf{B}$ is not the identity) hence $\mathbf{B}$ has order 3 or $ \mathbf{B} = 3$	A1	2.1
	(2)	

Notes

M1: Squares $\mathbf{B}$ to begin the process of finding the order, or uses a calculator to find $\mathbf{B}^3$

A1: Shows that $\mathbf{B}^3 = \mathbf{I}$ and concludes that order must be 3.

Mark scheme (b)

Scheme	Marks	AO
$\mathbf{I}$ {is the identity so} has order 1, and $\mathbf{E}$ {is $\mathbf{B}^{-1}$ so also} has order 3.	B1	1.1b
Finds at least one of the remaining orders e.g. $\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{= \mathbf{I}\}$ or $\mathbf{C}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{= \mathbf{I}\}$ or $\mathbf{D}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{= \mathbf{I}\}$	M1	1.1b
(So $\mathbf{I}$ has order 1, $\mathbf{B}$ and $\mathbf{E}$ order 3 and) $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ each have order 2	A1	1.1b
	(3)	

Notes

B1: States the order for $\mathbf{I}$ and gives the order for $\mathbf{E}$ (which is the inverse of $\mathbf{B}$).

M1: Finds the order of at least one of the other elements to get an element of order 2. May use a calculator

A1: Correct orders for the three elements of order 2.

(corrected from the printed mark scheme: the printed working shows $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$\mathbf{D}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, but $\mathbf{A}^2$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ directly and $\mathbf{D}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ before reducing modulo 2)

Mark scheme (c)

Scheme	Marks	AO
(i) There is no element of order 6 in G {so cannot be isomorphic to C_6 }	B1	2.4
(ii) Either • There are 12 symmetries of a regular hexagon • Hexagon has more than 6 symmetries • Hexagon has 2 rotations of order 6, could give an example • Hexagon has 2 rotations of order 3 • Hexagon has 1 rotation of order 2 • Hexagon has 6 reflections of order 2 • Group of symmetries of a hexagon has an order greater than 6 B0 for • The group of symmetries of a hexagon is also cyclic • The groups are not of the same order • Any incorrect orders of element stated • Comments on the order of the whole group • The group of symmetries has order 6	B1	2.4
	(2)	

Notes

B1: Correct reason given, identifies there is no element of **order 6**. It is not sufficient to say that it does not have a group generator, they need to give a reason.

B1: Any correct reason given.

Considers the total number of elements of the group, order of the group is more than 6

Considers the orders of either reflections or rotations

Mark scheme (d)

Scheme	Marks	AO
Any two correct matchings	M1	2.2a

Completes the matching with at least 4 correct	M1	3.1a																																																
Hence an isomorphism is	A1	2.1																																																
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Notes

M1: Any two correct matchings

M1: Completes the matching with at least 4 correct

A1: A fully specified correct isomorphism.