

QQQ – Further Pure 2 – Chapter 1 – Number Theory (Version 1)

Total Marks: 24

(24 = Platinum, 22 = Gold, 19 = Silver, 16 = Bronze)

1.

In this question you must show detailed reasoning.

Use Fermat's Little Theorem to determine the least positive residue of

$$21^{80} \pmod{23}$$

(Total 4 marks)

Edexcel A2 June 2024 Q1

2.

(i) Explain why the congruence equation

$$21x \equiv 11 \pmod{36}$$

has no solutions.

(2)

(ii) Use modular arithmetic to solve the congruence equation

$$36y \equiv 192 \pmod{40}$$

(4)

(Total 6 marks)

Edexcel A2 June 2025 Q7

3. The highest common factor of 963 and 657 is c .

(a) Use the Euclidean algorithm to find the value of c .

(3)

(b) Hence find integers a and b such that

$$963a + 657b = c$$

(3)

(Total 6 marks)

Edexcel AS October 2020 Q2

4.

- (i) The polynomial $F(x)$ is a quartic such that

$$F(x) = px^4 + qx^3 + 2x^2 + rx + s$$

where p, q, r and s are distinct constants.

Determine the number of possible quartics given that

- (a) the constants p, q, r and s belong to the set $\{-4, -2, 1, 3, 5\}$ (1)

- (b) the constants p, q, r and s belong to the set $\{-4, -2, 0, 1, 3, 5\}$ (1)

- (ii) A 3-digit positive integer $N = abc$ has the following properties

- N is divisible by 11
- the sum of the digits of N is even
- $N \equiv 8 \pmod{9}$

- (a) Use the first two properties to show that

$$a - b + c = 0$$

(2)

- (b) Hence determine all possible integers N , showing all your working and reasoning.

(4)

(Total 8 marks)

Solutions

1.

Scheme	Marks	AO
$21^{22} \equiv 1 \pmod{23}$ or $21^{23} \equiv 21 \pmod{23}$ or $21^{23} \equiv -2 \pmod{23}$	B1	1.2
$80 = 3 \times 22 + 14 \Rightarrow 21^{80} \equiv 21^{14} \pmod{23}$ $80 = 3 \times 23 + 11 \Rightarrow 21^{80} \equiv -8 \times 21^{11} \pmod{23}$	M1	1.1b
For example $\equiv ((-2)^4)^3 \times (-2)^2 \equiv 16^3 \times 4 \equiv (-7)^2 \times -7 \times 4 \equiv 49 \times -28 \equiv 3 \times 18 \equiv 54$ $\equiv (-2)^{14} \pmod{23} \equiv 16384 \pmod{23}$ $\equiv (-2)^{14} \equiv (4)^7 \pmod{23}$ $\equiv ((-2)^4)^3 \times (-2)^2 \equiv 16^3 \times 4 \equiv \{16384\}$ $21^2 = 441 \equiv 4 \pmod{23} \Rightarrow (21^2)^7 = (4)^7 = \{16384\}$ $(3 \times 7)^{14} = 3^{14} \times 7^{14} = 3^{14} \times (49)^7 \equiv 4 \times (3)^7$ $\pmod{23} \equiv 4 \times 2 \pmod{23}$ $\equiv -8 \times (-2)^{11} \equiv 16 \times 2^{10}$ Leading to $\equiv \dots \pmod{23}$	dM1	1.1b
$\equiv 8 \pmod{23}$	A1	2.2a
	(4)	
	(4 marks)	

Notes

B1: Accurately recalls Fermat's Little Theorem. Allow if stated correctly in general form, $a^{p-1} \equiv 1 \pmod{p}$. May be implied.

M1: Attempts to write the index in a suitable form and then applies the theorem, look for $80 = 22k + r$ or equivalent leading to a reduced index.

dM1: For a full method to reduce to a least residue. There are many ways that they can split up their value, including use of a calculator to work with a higher power as long as an attempt at FLT to reduce initially has been seen. Condone a sign slip e.g using $21^2 \equiv 19 \pmod{23}$ instead of $21^2 \equiv -19 \pmod{23}$ as long as there is a complete method and there is only one sign slip

A1: Correct answer.

2.

Mark scheme (i)

Scheme	Marks	AO
$\gcd(36, 21) = 3$	M1	1.1b
3 does not divide into 11 therefore no solutions	A1	2.4
	(2)	

Notes

M1: Finds the gcd of 36 and 21 or in some way considers the hcf of these two numbers.

A1: States that the gcd is 3 and does not divide into 11 therefore no solutions

Mark scheme (ii)

Scheme	Marks	AO
$12 \times 3y \equiv 12 \times 16 \pmod{40}$	M1	1.1b
$\gcd(12, 40) = 4$ and divides through by 4	M1	1.1b
$3y \equiv 16 \pmod{10} \Rightarrow 3y \equiv 6 \pmod{10}$ $\gcd(3, 10) = 1$	dM1	2.1
$y \equiv 2 \pmod{10}$ or $y \equiv 2, 12, 22, 32 \pmod{40}$	A1	2.2a
	(4)	
	(6 marks)	

Notes

M1: Identifies $36y$ as kay and 192 as kb (may be implied by cancelling).

M1: Finds the gcd of k and 40. Then divides through by the gcd of k and 40, Modulus must be reduced to 10.

dM1: Completes the solution by appropriate means. There will be many variations on how they can do this. Accept just $y = 2$ as an attempt to find the solution. Note the multiplicative inverse of 9 modulo 10 is 9. If a multiplicative inverse is attempted, it must be from a correct method.

A1: Deduces the correct solution

Alternative I

Scheme	Marks	AO
$36y \equiv 192 \equiv 32 \pmod{40}$	M1	1.1b
$\text{hcf}(36, 40) = 4 \Rightarrow 9y \equiv 8 \pmod{10}$	M1	1.1b
E.g. $9y \equiv 18 \pmod{10} \Rightarrow y \equiv 2 \pmod{10}$	dM1	2.1
$y \equiv 2 \pmod{10}$ or $y \equiv 2, 12, 22, 32 \pmod{40}$	A1	2.2a
	(4)	

M1: Reduces RHS modulo 40 to give $36y \equiv 32 \pmod{40}$

M1: Finds the gcd of 36 and 40 and divides through by 4. This must include the modulus being reduced to 10.

dM1: Depends on previous M. Finds a value for y by any appropriate means. There will be many variations on how they can do this. Accept just $y = 2$ as an attempt to find the solution.

A1: Deduces the correct solution

Note: Candidates to start by finding the hcf of 36 and 40 and divide through to get $9y \equiv 48 \pmod{10}$ will score M2 at the start of the solution.

(ii) Alt II

Scheme	Marks	AO
$36y \equiv 192 \pmod{40} \Rightarrow 36y = 192 + 40k$	M1	1.1b
$\Rightarrow 9y = 48 + 10k$	M1	1.1b
$\Rightarrow 10y - y = 50 - 2 + 10k \Rightarrow y = 10y - 50 - 10k + 2$ $\Rightarrow y = 2 + 10(y - 5 - k)$	dM1	2.1
$y \equiv 2 \pmod{10}$ or $y \equiv 2, 12, 22, 32 \pmod{40}$	A1	2.2a
	(4)	

M1: Writes the congruence as a linear equation with $40k$

M1: Divides the equation through by 4

dM1: Completes the solution by appropriate means. One example shown in scheme.

A1: Deduces the correct solution.

3.

Mark scheme (a)

Scheme	Marks	AO
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$963 = 657 \times 1 + 306$ $657 = 306 \times 2 + 45$	M1	1.2
$306 = 45 \times 6 + 36$ $45 = 36 \times 1 + 9$ $36 = 9 \times 4 + 0$	A1	1.1b
$\text{HCF}(963, 657) = 9$ or $c = 9$	A1	1.1b
	(3)	

Notes

M1: Starts the process by showing 2 correct stages

A1: Completes the algorithm correctly

A1: Correct HCF

Mark scheme (b)

Scheme	Marks	AO
$9 = 45 - 36 \times 1$ $9 = 45 - (306 - 45 \times 6)$	M1	1.1b
$9 = 7 \times 45 - 306$ $= 7 \times (657 - 2 \times 306) - 306 = 7 \times 657 - 15 \times 306$ $= 7 \times 657 - 15 \times (963 - 1 \times 657)$	A1	2.1
$9 = -15 \times 963 + 22 \times 657$	A1	1.1b
	(3)	
	(6 marks)	

Notes

M1: Starts the reversal process by completing at least 2 stages

A1: Completes the algorithm correctly

A1: Correct values for a and b

4.

Mark scheme (i)(a)

Scheme	Marks	AO
120	B1	1.1b

	(1)	
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Notes

B1: See scheme

Mark scheme (i)(b)

Scheme	Marks	AO
300	B1	1.1b
	(1)	

Notes

B1: See scheme

Mark scheme (ii)(a)

Scheme	Marks	AO
As divisible by 11 implies $a - b + c = 11p$ where p is an integer Due to the restrictions on a, b and c $\Rightarrow a - b + c = 11$ or 0	M1	2.4
For example $a + b + c$ is even $\Rightarrow (a + b + c) - (2b)$ is even $\Rightarrow a - b + c$ is even or Sum of digits is even implies that $a + b + c = 2q$ $(a - b + c = (a + b + c) - 2b = 2q - 2b = 2n)$ Alternative approach 1 $a + b + c = 2q$ where $q \in \mathbb{Z}$ either $a + b + c = 2q$ and $a - b + c = 0 \Rightarrow 2b = 2q \Rightarrow b = q$ valid $a + b + c = 2q$ and $a - b + c = 11 \Rightarrow 2b = 2q - 11 \Rightarrow b = q - 5.5$ not valid Or $a + b + c = 2q$ and $a - b + c = 0 \Rightarrow 2a + 2c = 2q$ valid	A1*	2.1

$a + b + c = 2q$ and $a - b + c = 11 \Rightarrow 2a + 2c = 2q + 11$ not valid as $2(a + c)$ even or This approach may be in words Alternative approach 2 If sum $a - b + c$ is odd/11 then either one or all three numbers odd. This would mean sum $a + b + c$ would be odd/not even Alternative approach 3 If $a - b + c = 11$ this implies $a + b + c \equiv 1 \pmod{2}$ (contradiction) <u>Conclusion</u> therefore $a - b + c = 0 \pmod{11}$		
	(2)	

Notes

M1: Uses the divisibility rule for 11 and the restrictions on the values a, b and c leading to $a - b + c = 0$ or 11 only

A1*: Uses the information that the sum of the digits is even and that $a - b + c = 0$ or 11 from the divisibility rule for 11 to **show** that $a - b + c = 0 \pmod{11}$

Mark scheme (ii)(b)

Scheme	Marks	AO
$N = 100a + 10b + c \Rightarrow a + b + c \equiv 8 \pmod{9}$ o.e. or $N + 1 \equiv 0 \pmod{9}$ ($N + 1$ is a multiple of 9) o.e. $\Rightarrow a + b + c + 1 \equiv 0 \pmod{9}$ ($a + b + c + 1$ is a multiple of 9) o.e	B1	1.1b
Solves $a - b + c = 0$ and $a + b + c \equiv 8 \pmod{9}$ either ($a + b + c \equiv 8$ or 26) to find a value for b ($= 4$) and uses this value to form an equation $a + c = \dots$ (4)	M1	2.1

Or $2a + 2c \equiv 8 \pmod{9} \Rightarrow a + c \equiv 4 \pmod{9}$ leading to $a + c = \dots (4)$ or $2a + 2c \equiv 8 \pmod{9} \Rightarrow a + c \equiv 4 \pmod{9} \Rightarrow b \equiv 4 \pmod{9}$ leading to $b = \dots (4)$ Or $a - b + c = 0$, $b = a + c$ so $2b + 1$ is a multiple of 9, $2b + 1 = 9$ or 18 leading to $b = \dots (4)$		
$N = 143, 242, 341, 440$	M1 A1	1.1b 2.2a
	(4)	
	(8 marks)	

Notes

B1: States $a + b + c \equiv 8 \pmod{9}$ or $a + b + c + 1 \equiv 0 \pmod{9}$

M1: Uses their equations to find a value for $a + c = \dots$ or a value for b

M1: Uses their values to find at least two correct values for N

A1: Deduces all four correct values of N and no extra values

Trial and error

B1: For identifying $a + c = b$

M1: For one correct value, this implies B1

M1: For two correct values

A1: Deduces all four correct values of N and no extra values

Alternative

Scheme	Marks	AO
$(N) = 11n \equiv 8 \pmod{9}$	B1	1.1b
Finds multiplicative inverse of 11 using $5 \times 9 - 4 \times 11 = 1$ $-4 \times 11n \equiv -4 \times 8 \pmod{9} \Rightarrow n \equiv -32 \pmod{9} \Rightarrow n \equiv 4 \pmod{9}$ So $N = 11(9n + 4)$	M1	2.1
$N = 143, 242, 341, 440$	M1	1.1b

	A1	2.2a
	(4)	

B1: States $(N) = 11n \equiv 8 \pmod{9}$

M1: Finds the multiplicative inverse of 11 and uses this to find an expression for N

M1: Uses their expression for N to find at least two values for N

A1: Deduces all four correct values of N and no extra values