

QQQ – Further Pure 1 – Chapter 9 – Reducible Differential Equations (Version 2)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. The concentration, $P \text{ mg m}^{-3}$, of a pollutant in a reservoir, t days after the pollutant entered the reservoir, is modelled by the differential equation

$$\frac{1}{P} \frac{dP}{dt} = 4tP^2 - 1 \quad (\text{I})$$

- (a) Show that the transformation $x = \frac{1}{P^2}$ transforms equation (I) into the equation

$$\frac{dx}{dt} - 2x = -8t \quad (\text{II})$$

(3)

Given that $P = 0.5$ when $t = 0$

- (b) solve differential equation (II) to show that, according to the model,

$$P^2 = \frac{1}{4t + 2 + ke^{2t}}$$

where k is a constant to be determined.

(6)

Given that the concentration of the pollutant in the reservoir, 3 days after the pollutant entered the reservoir, was 0.034 mg m^{-3}

- (c) comment on the reliability of the model, giving a reason for your answer.

(2)

(Total 11 marks)

2.

- (a)** Show that the transformation $y = xv$ transforms the equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + (2 + 9x^2)y = x^5, \quad \text{I}$$

into the equation

$$\frac{d^2 v}{dx^2} + 9v = x^2. \quad \text{II}$$

(5)

- (b)** Solve the differential equation II to find v as a function of x .

(6)

- (c)** Hence state the general solution of the differential equation I.

(1)

(Total 12 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
Examples: $x = \frac{1}{P^2} \Rightarrow \frac{dx}{dt} = -\frac{2}{P^3} \frac{dP}{dt}$ or $\frac{dP}{dt} = \frac{dP}{dx} \times \frac{dx}{dt} = -\frac{P^3}{2} \frac{dx}{dt}$ or $P = x^{-\frac{1}{2}} \Rightarrow \frac{dP}{dt} = -\frac{1}{2} x^{-\frac{3}{2}} \frac{dx}{dt}$ or $\frac{dP}{dx} = -\frac{1}{2} x^{-\frac{3}{2}} \Rightarrow \frac{dx}{dt} = \frac{dx}{dP} \times \frac{dP}{dt} = -2x^{\frac{3}{2}} \times (4tP^3 - P)$ or $x = \frac{1}{P^2} \Rightarrow xP^2 = 1 \Rightarrow P^2 \frac{dx}{dt} + 2Px \frac{dP}{dt} = 0$ or $x = \frac{1}{P^2} \Rightarrow P^2 = \frac{1}{x} \Rightarrow 2P \frac{dP}{dt} = -\frac{1}{x^2} \frac{dx}{dt}$	M1	3.1a
$\frac{1}{P} \times -\frac{P^3}{2} \frac{dx}{dt} = \frac{4t}{x} - 1 \Rightarrow -\frac{1}{2x} \frac{dx}{dt} = \frac{4t}{x} - 1$ or $x^{\frac{1}{2}} \times -\frac{1}{2} x^{-\frac{3}{2}} \frac{dx}{dt} = \frac{4t}{x} - 1 \Rightarrow -\frac{1}{2x} \frac{dx}{dt} = \frac{4t}{x} - 1$ or $\frac{dx}{dt} = -2x^{\frac{3}{2}} \times (4tP^3 - P) = -2x^{\frac{3}{2}} \times \left(\frac{4t}{x^{\frac{3}{2}}} - \frac{1}{x^{\frac{1}{2}}} \right)$ or $\frac{1}{P} \times -\frac{P}{2x} \frac{dx}{dt} = \frac{4t}{x} - 1 \Rightarrow -\frac{1}{2x} \frac{dx}{dt} = \frac{4t}{x} - 1$ or $\frac{1}{P} \times -\frac{1}{2Px^2} \frac{dx}{dt} = \frac{4t}{x} - 1 \Rightarrow -\frac{1}{2x} \frac{dx}{dt} = \frac{4t}{x} - 1$	M1	1.1b

$\frac{dx}{dt} - 2x = -8t * \text{cso}$	A1*	2.1
	(3)	

Notes

M1: Identifies and applies a correct strategy for the differentiation. This may be seen when $\frac{dP}{dt}$ is substituted into the equation.

M1: Substitutes into the given differential equation and proceeds to an equation in x and t only.

A1*: Correct proof with sufficient working shown and no errors, cso

Mark scheme (b)

Scheme	Marks	AO
$I = e^{\int -2 dt} = e^{-2t}$	B1	2.2a
$xe^{-2t} = \int -8te^{-2t} \{dt\}$	M1	1.1b
$= 4te^{-2t} - \int 4e^{-2t} \{dt\}$	M1	3.1a
$= 4te^{-2t} + 2e^{-2t} + k$	A1	1.1b
$xe^{-2t} = 4te^{-2t} + 2e^{-2t} + k \Rightarrow \frac{1}{0.5^2} = 2 + k \Rightarrow k = \dots \{2\}$ or $xe^{-2t} = 4te^{-2t} + 2e^{-2t} + k \Rightarrow x = 4t + 2 + ke^{2t} \Rightarrow \frac{1}{P^2} = 4t + 2 + ke^{2t}$ $P = 0.5, t = 0 \Rightarrow \frac{1}{0.5^2} = 2 + k \Rightarrow k = \dots \{2\}$	M1	3.4
$\frac{1}{P^2} = 4t + 2 + 2e^{2t} \Rightarrow P^2 = \frac{1}{4t + 2 + 2e^{2t}} \text{ cso}$	A1	2.1
	(6)	

Notes

B1: Deduces the correct integrating factor

M1: Applies their integrating factor I to obtain $Ix = \int \pm 8It \{dt\}$ condone missing dt

M1: Recognises that integration by parts is required and applies this correctly to reach the form

$$= Ate^{-2t} - \int Be^{-2t} \{dt\}$$

A1: Correct integration including an arbitrary constant

M1: Uses the conditions given in the model to find the constant of integration. This may be seen before rearranging to get $P^2 = \dots$, condone a slip

A1: Correct equation with no errors seen, cso, missing dt during working would lose this mark

Note that the first 4 marks in (b) can also be obtained as follows:

B1: AE: $m - 2 = 0 \Rightarrow m = 2 \Rightarrow x = Ae^{2t}$ (Correct CF)

M1: PI: $x = at + b \Rightarrow \frac{dx}{dt} = a$ (Selects the correct PI form and differentiates)

M1: $a - 2at - 2b = -8t \Rightarrow a = \dots (4), b = \dots (2)$ (Substitutes and compares coefficients to find a and b)

A1: $x = 4t + 2 + Ae^{2t}$ (Correct expression)

Mark scheme (c)

Scheme	Marks	A0
$t = 3 \Rightarrow P^2 = \frac{1}{14 + 2e^6} \Rightarrow P = 0.03490326 \dots$	M1	3.4
This value is close to 0.034 so the model is reliable. - Between 0.0323 and 0.0357 good model - Between 0.0306 and 0.0323 or 0.0357 and 0.0374 either a good or bad model with reasoning - Less than 0.0306 or more than 0.0374 bad model	A1ft	3.5a
	(2)	
	(11 marks)	

Notes

M1: Uses the model with $t = 3$ to find P , where $P^2 > 0$ or compares their value of P^2 with $0.001156 = 0.034^2$

A1ft: Compares their value of P with 0.034 and makes a suitable conclusion, making any comparison with P^2 is A0

2.

Mark scheme (a)

Scheme	Marks
$\frac{dy}{dx} = x \frac{dv}{dx} + v, \quad \frac{d^2y}{dx^2} = x \frac{d^2v}{dx^2} + 2 \frac{dv}{dx}$ <i>[M1 for diff. product, A1 both correct]</i>	M1A1
$\therefore x^2 \left(x \frac{d^2v}{dx^2} + 2 \frac{dv}{dx} \right) - 2x \left(x \frac{dv}{dx} + v \right) + (2 + 9x^2)vx = x^5$	M1
$x^3 \frac{d^2v}{dx^2} + 2x^2 \frac{dv}{dx} - 2x^2 \frac{dv}{dx} - 2vx + 2vx + 9vx^3 = x^5$ $\left[x^3 \frac{d^2v}{dx^2} + 9vx^3 = x^5 \right]$	A1
Given result: $\frac{d^2v}{dx^2} + 9v = x^2$ <i>cso</i>	A1
	(5)

Mark scheme (b)

Scheme	Marks
CF: $v = A \sin 3x + b \cos 3x$ (may just write it down)	M1A1
Appropriate form for PI: $v = \lambda x^2 + \mu$ (or $ax^2 + bx + c$)	M1
Complete method to find λ and μ (or a, b, c)	M1
$v = A \sin 3x + B \cos 3x + \frac{1}{9}x^2 - \frac{2}{81}$ <i>[f.t. only on wrong CF]</i>	M1A1ft
	(6)

Mark scheme (c)

Scheme	Marks
$\therefore y = Ax \sin 3x + Bx \cos 3x + \frac{1}{9}x^3 - \frac{2}{81}x$ <i>[f.t. for $y = x$ (candidate's CF + PI), providing two arbitrary constants]</i>	B1ft
	(1)

	(12 marks)
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