

QQQ – Further Pure 1 – Chapter 9 – Reducible Differential Equations (Version 1)

Total Marks: 25

(25 = Platinum, 23 = Gold, 20 = Silver, 17 = Bronze)

1.

- (a)** Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x}, \quad x > 0, \quad y > 0 \quad (\text{I})$$

into the differential equation

$$x \frac{dv}{dx} = 2v + \frac{1}{v}. \quad (\text{II})$$

(3)

- (b)** By solving differential equation (II), find a general solution of differential equation (I) in the form $y = f(x)$. **(7)**

Given that $y = 3$ at $x = 1$,

- (c)** find the particular solution of differential equation (I). **(2)**

(Total 12 marks)

2. A particle P moves along a straight line.

At time t minutes, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t^2 \frac{d^2x}{dt^2} - 2t \frac{dx}{dt} + 2x + 16t^2x = 4t^3 \sin 2t \quad (\text{I})$$

(a) Show that the transformation $x = ty$ transforms equation (I) into the equation

$$\frac{d^2y}{dt^2} + 16y = 4 \sin 2t$$

(5)

(b) Hence find a general solution for the displacement of P from O at time t minutes.

(8)

(Total 13 marks)

Solutions

1.

Mark scheme (a)

Scheme	Marks
$\frac{dy}{dx} = v + x \frac{dv}{dx}$	B1
$\left(v + x \frac{dv}{dx}\right) = \frac{x}{vx} + \frac{3vx}{x} \Rightarrow x \frac{dv}{dx} = 2v + \frac{1}{v} (*)$	M1A1
	(3)

Notes

B1 for statement printed or for $\frac{dy}{dx} = \left(x + v \frac{dx}{dv}\right) \frac{dv}{dx}$

First M1 is for RHS of equation only but for A1 need whole answer correct.

Mark scheme (b)

Scheme	Marks
$\int \frac{v}{1 + 2v^2} dv = \int \frac{1}{x} dx$	M1
$\frac{1}{4} \ln(1 + 2v^2), = \ln x (+C)$	dM1A1, B1
$Ax^4 = 1 + 2v^2$	dM1
$Ax^4 = 1 + 2\left(\frac{y}{x}\right)^2$ so $y = \sqrt{\frac{Ax^6 - x^2}{2}}$ or $y = x\sqrt{\frac{Ax^4 - 1}{2}}$ or $y = x\sqrt{\left(\frac{1}{2}e^{4\ln x + 4c} - \frac{1}{2}\right)}$	M1A1
	(7)

Notes

First M1 accept $\int \frac{1}{2v + \frac{1}{v}} dv = \int \frac{1}{x} dx$

Second M1 requires an integration of correct form $\frac{1}{4}$ may be missing

A1 for LHS correct with $\frac{1}{4}$ and B1 is independent and is for $\ln x$

Third M1 is **dependent** and needs correct application of log laws

Fourth M1 is independent and merely requires return to y/x for v

N.B. There is an IF method possible after suitable rearrangement – see note.

Mark scheme (c)

Scheme	Marks
$x = 1$ at $y = 3: 3 = \sqrt{\frac{A-1}{2}} \quad A = \dots$	M1
$y = \sqrt{\frac{19x^6 - x^2}{2}}$ or $y = x\sqrt{\frac{19x^4 - 1}{2}}$	A1
	(2)
	(12 marks)

2.

Mark scheme (a)

Scheme	Marks	AO
Use of $x = ty$ to give $\frac{dx}{dt} = y + t \frac{dy}{dt}$ or $y = \frac{x}{t} \rightarrow \frac{dy}{dt} = -\frac{x}{t^2} + \frac{1}{t} \frac{dx}{dt}$ oe	B1	1.1b
$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dy}{dt} + t \frac{d^2y}{dt^2}$ or $\frac{d^2y}{dt^2} = -\frac{1}{t^2} \frac{dx}{dt} + \frac{2x}{t^3} + \frac{1}{t} \frac{d^2x}{dt^2} - \frac{1}{t^2} \frac{dx}{dt}$ oe	M1 A1	2.1 1.1b
$t^2 \left[\text{their } \frac{d^2x}{dt^2} \right] - 2t \left[\text{their } \frac{dx}{dt} \right] + 2[ty] + 16t^2[ty] =$ $4t^3 \sin 2t$ Or $\left[\text{their } \frac{d^2y}{dt^2} \right] + 16\frac{x}{t} = 4 \sin 2t$	dM1	2.1
$t^2 \left[\frac{dy}{dt} + \frac{dy}{dt} + t \frac{d^2y}{dt^2} \right] - 2t \left[y + t \frac{dy}{dt} \right] + 2[ty] + 16t^2[ty] =$ $4t^3 \sin 2t$ $t^3 \frac{d^2y}{dt^2} + 16t^3y = 4t^3 \sin 2t \Rightarrow \frac{d^2y}{dt^2} + 16y = 4 \sin 2t$ * (oe in	A1*	1.1b

reverse)		
	(5)	

Notes

B1: For a correct suitable first derivative expression linking $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

M1: Uses the product rule to find an equation linking second derivatives from their first derivative expression.

A1: A correct second derivative expression.

dM1: Substitutes the first and second derivatives and replaces x with ty to obtain a differential equation in y and t only. Alternatively, may go in reverse and replace y with $\frac{x}{t}$ etc in the second equation to obtain a differential equation in x and t only.

A1*: Simplifies their expression with a correct intermediate stage/working to reach the printed answer. Alternatively, correct working in the other direction to achieve equation (I) from the final equation.

Mark scheme (b)

Scheme	Marks	AO
Solves $m^2 + 16 = 0$ to give $m = \dots$	M1	1.1b
$(y =) A \cos 4t + B \sin 4t$	A1	1.1b
Particular integral $(y =) \underline{\lambda \sin 2t} + \mu \cos 2t$	B1	2.2a
$\frac{dy}{dt} = 2\lambda \cos 2t - 2\mu \sin 2t$ and $\frac{d^2y}{dt^2} = -4\lambda \sin 2t - 4\mu \cos 2t$	M1	1.1b
Substitutes into the differential equation and finds values for λ and μ $[-4\lambda \sin 2t - 4\mu \cos 2t] + 16[\lambda \sin 2t + \mu \cos 2t] = 4 \sin 2t$ $\Rightarrow \lambda = \dots, \mu = \dots$	dM1	2.1
$y = "A \cos 4t + B \sin 4t" + \frac{1}{3} \sin 2t$	A1ft	1.1b
$x = t[\text{their } y]$	M1	3.4
$x = t \left[A \cos 4t + B \sin 4t + \frac{1}{3} \sin 2t \right]$	A1	2.2a

	(8)	
	(13 marks)	

Notes

M1: Forms the correct auxiliary equation and attempts to solve (any values after the correct AE seen)

A1: Correct complementary function. Accept for this mark if they give it in terms of x - you are looking for the correct form for the CF.

B1: Deduces a correct form of the particular integral (must include at least $\lambda \sin 2t$ but may be no more than this). SC if by error the CF includes $\sin 2t$ allow B1 for a PI of form $\lambda t \sin 2t + \mu t \cos 2t$

M1: Differentiates the PI twice.

dM1: Dependent on the previous method mark. Substitutes y and $\frac{d^2y}{dt^2}$ into the differential equation leading to values for the constant(s).

A1ft: Correct general equation for y following through their CF, which must be a (non-constant) function of t . Must be in terms of t and start $y = \dots$

M1: Links the solution to the solution of the model equation to find the general solution for the displacement.

A1: Deduces the correct general solution for the displacement. Must be $x = \dots$