

QQQ – Further Pure 1 – Chapter 7 – Methods in Calculus (Version 2)

Total Marks: 24

(24 = Platinum, 22 = Gold, 19 = Silver, 16 = Bronze)

1. Given that k is a real non-zero constant and that

$$y = x^3 \sin kx$$

use Leibnitz's theorem to show that

$$\frac{d^5 y}{dx^5} = (k^2 x^2 + A)k^3 x \cos kx + B(k^2 x^2 + C)k^2 \sin kx$$

where A , B and C are integers to be determined.

(4)

(Total 4 marks)

Edexcel A2 June 2019 Q2

2. Use l'Hospital's Rule to show that

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{(e^{\sin x} - \cos(3x) - e)}{\tan(2x)} = -\frac{3}{2}$$

(5)

(Total 5 marks)

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3.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

- (a) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to show that

$$\int \frac{1}{2 \sin \theta + \cos \theta + 2} d\theta = \int \frac{a}{(t+b)^2 + c} dt$$

where a , b and c are constants to be determined.

(3)

- (b) Hence show that

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2 \sin \theta + \cos \theta + 2} d\theta = \ln\left(\frac{2\sqrt{3}}{3}\right)$$

(4)

(Total 7 marks)

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4.

$$f(x) = x^4 \sin(2x)$$

Use Leibnitz's theorem to show that the coefficient of $(x - \pi)^8$ in the Taylor series expansion of $f(x)$ about π is

$$\frac{a\pi + b\pi^3}{315}$$

where a and b are integers to be determined.

(8)

$$\left[\begin{array}{l} \text{The Taylor series expansion of } f(x) \text{ about } x = k \text{ is given by} \\ f(x) = f(k) + (x - k)f'(k) + \frac{(x - k)^2}{2!}f''(k) + \dots + \frac{(x - k)^r}{r!}f^{(r)}(k) + \dots \end{array} \right]$$

(Total 8 marks)

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Solutions

1.

Scheme	Marks	AO
$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2, \frac{d^2u}{dx^2} = 6x, \frac{d^3u}{dx^3} = 6$	M1	1.1b
$v = \sin kx \Rightarrow \frac{dv}{dx} = k \cos kx, \frac{d^2v}{dx^2} = -k^2 \sin kx, \frac{d^3v}{dx^3} = -k^3 \cos kx,$ $\frac{d^4v}{dx^4} = k^4 \sin kx, \frac{d^5v}{dx^5} = k^5 \cos kx$	M1	2.1
$\frac{d^5y}{dx^5} = x^3 k^5 \cos kx + 5 \times 3x^2 \times k^4 \sin kx + \frac{5 \times 4}{2} \times 6x \times (-k^3 \cos kx) +$ $\frac{5 \times 4 \times 3}{3!} \times 6 \times (-k^2 \sin kx)$	M1	2.1
$= (k^2 x^2 - 60) k^3 x \cos kx + 15 (k^2 x^2 - 4) k^2 \sin kx$	A1	1.1b
	(4)	
	(4 marks)	

Notes

M1: Differentiates $u = x^3$ three times. Need to see $x^3 \rightarrow \dots x^2 \rightarrow \dots x \rightarrow k$

M1: Uses $v = \sin kx$ to establish the form of the derivatives. Need to see at least alternating $k^{\dots} \sin kx$ and $k^{\dots} \cos kx$ with increasing powers of k for at least 3 derivatives.

M1: Uses a correct formula with 2 and 3! (or 6) with terms shown to disappear after the fourth term. This needs to be a correct application of the theorem so that the correct binomial coefficients need to go with the correct pairings of their derivatives. If there is any doubt, at least 3 terms should have the correct structure. Allow equivalent notation for the binomial coefficients e.g. $\binom{5}{0}, \binom{5}{1}$ etc. or ${}^5C_0, {}^5C_1$ etc.

A1: Correct expression in the required form with correct values of A, B and C . Apply isw if necessary e.g. if a correct expression is followed by $A = 60, B = 15, C = -4$
(NB $A = -60, B = 15, C = -4$)

If there is no use Leibnitz's theorem e.g. repeated differentiation of products, this scores no marks.

2.

Scheme	Marks	AO
$\frac{\frac{d}{dx}(e^{\sin x} - \cos(3x) - e)}{\frac{d}{dx}(\tan(2x))} = \frac{\pm \cos(x)e^{\sin x} \pm A \sin(3x)}{B \sec^2 2x}$	M1	1.1b

$\frac{\frac{d}{dx}(e^{\sin x} - \cos(3x) - e)}{\frac{d}{dx}(\tan(2x))} = \frac{\cos(x)e^{\sin x} + 3 \sin(3x)}{2 \sec^2 2x}$	A1	1.1b
$\frac{\cos(x)e^{\sin x} + 3 \sin(3x)}{2 \sec^2 2x}$	A1	1.1b
$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos(x)e^{\sin x} + 3 \sin(3x)}{2 \sec^2 2x}$ $= \frac{\cos(\frac{\pi}{2})e^{\sin(\frac{\pi}{2})} + 3 \sin(\frac{3\pi}{2})}{2 \sec^2(\frac{2\pi}{2})}$ or $= \frac{0 \times e + 3 \times (-1)}{2 \times (-1)^2} = \dots$	M1	1.2
$= -\frac{3}{2} *$	A1*	2.1
	(5)	
	(5 marks)	

Notes

M1: Attempts differentiation of both numerator and denominator, including at least one use of the chain rule. Either numerator or denominator of the correct form. May be done separately.

A1: Numerator correct

A1: Denominator correct

M1: Applies l'Hospital's Rule, must **see clear use of a substitution** of $x = \frac{\pi}{2}$ into their derivatives, not the original expression. (no need to see check that limits of numerator and denominator are non-zero).

A1*: Needs to be a correct intermediate line following substitution before reaching the printed answer with use of some limit notation. All aspects of the proof should be clear for this mark to be awarded and no errors seen.

3.

Mark scheme (a)

Scheme	Marks	AO
$\int \frac{1}{2 \sin \theta + \cos \theta + 2} d\theta = \int \frac{1}{2 \left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) + 2} \times \frac{2}{1+t^2} dt$	M1	2.1
$\int \frac{2}{4t + (1 - t^2) + 2(1 + t^2)} dt = \int \frac{2}{t^2 + 4t + 3} dt$	M1	1.1b
$\int \frac{2}{(t + 2)^2 - 1} dt$	A1	2.2a
	(3)	

Notes

M1: Applies the correct substitutions $\sin \theta = \frac{2t}{1+t^2}$, $\cos \theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2}{1+t^2} dt$ The dt may be missing for this and the next M.

M1: Simplifies the integrand to the form $\frac{a}{pt^2 + qt + r}$ allowing slips in coefficients but the “ $1 + t^2$ ” must be correctly dealt with.

A1: Correct answer in the form required, including the $\int \dots dt$

Mark scheme (b)

Scheme	Marks	AO
$\int \frac{a}{(t+b)^2 - c} dt = \frac{a}{2\sqrt{c}} \ln \left \frac{(t+b) - \sqrt{c}}{(t+b) + \sqrt{c}} \right , c > 0 \text{ or}$ $\int \frac{a}{(t+b)^2 + c} dt = \frac{a}{\sqrt{c}} \arctan \left(\frac{t+b}{\sqrt{c}} \right), c > 0$ Note correct answer is $\int \frac{2}{(t+2)^2 - 1} dt = \frac{2}{2(1)} \ln \left \frac{(t+2) - 1}{(t+2) + 1} \right$	M1	3.1a
Correctly uses the limits $t = 1$ and $t = \sqrt{3} \rightarrow \ln \left \frac{\sqrt{3} + 1}{\sqrt{3} + 3} \right - \ln \left \frac{2}{4} \right$ if correct.	M1	1.1b
$= \ln \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 3} \div \frac{1}{2} \right)$	M1	1.1b
$= \ln \left(2 \times \frac{\sqrt{3} + 1}{\sqrt{3} + 3} \times \frac{\sqrt{3} - 3}{\sqrt{3} - 3} \right) = \ln \left(\frac{2\sqrt{3}}{3} \right) *$	A1*	2.1
	(4)	
	(7 marks)	

Notes

M1: A correct method for the integration, by use of standard formula. Modulus signs are not necessary. Note that $-\frac{a}{\sqrt{c}} \operatorname{artanh} \left(\frac{t+b}{\sqrt{c}} \right) = -\frac{a}{2\sqrt{c}} \ln \left| \frac{t+b+\sqrt{c}}{t+b-\sqrt{c}} \right|$ is also a correct version. If by error they end up with a positive c then accept for the arctan integral shown in scheme.

M1: Correctly uses the limits $t = 1$ and $t = \sqrt{3}$ in their integral to produce an exact expression. Use of the original limits is M0, must have changed to t (unless they reverse the substitution).

M1: Correctly combines the ln terms with at least one step showing the result of substituting limits before the final answer, and their final answer must follow this step if no further working is shown. May be seen before or after the rationalisation. (M0 if no ln terms.)

A1* cso: Shows the method to rationalise the denominator and correctly completes to the given answer. Some suitable working must be seen. Note that $\sqrt{3} + 3 = \sqrt{3}(1 + \sqrt{3})$ may be used in the denominator, followed by cancelling.

(b) Alt

Scheme	Marks	AO
$\int \frac{2}{(t+2)^2 - 1} dt = \int \frac{2}{(t+1)(t+3)} dt$ $= \int \frac{1}{t+1} - \frac{1}{t+3} dt = \ln t+1 - \ln t+3 $	M1	3.1a
Correctly uses the limits $t = 1$ and $t = \sqrt{3}$ $\ln(\sqrt{3}+1) - \ln(\sqrt{3}+3) - \ln 2 + \ln 4$	M1	1.1b
$= \ln\left(2 \times \frac{\sqrt{3}+1}{\sqrt{3}+3}\right)$	M1	2.1
$= \ln\left(2 \times \frac{\sqrt{3}+1}{\sqrt{3}(1+\sqrt{3})}\right) = \ln\left(\frac{2\sqrt{3}}{3}\right) *$	A1*	1.1b
	(4)	

M1: A correct method for the integration, applying partial fractions and integrating to ln terms.

M1: Correctly uses the limits $t = 1$ and $t = \sqrt{3}$ (or correct θ limits if substitution is reversed) to produce an exact expression.

M1: Correctly combines the ln terms with at least one step showing the result of substituting limits before the final answer, and their final answer must follow this step if no further working is shown. May be seen before or after the rationalisation. (M0 if no ln terms.)

A1* cso: Shows the method to rationalise the denominator and correctly completes to the given answer. Some suitable working must be seen. Note that $\sqrt{3} + 3 = \sqrt{3}(1 + \sqrt{3})$ may be used in the denominator, followed by cancelling.

4.

Scheme	Marks	AO
$f(x) = x^4 \sin(2x) \quad u = x^4 \quad v = \sin(2x)$		
$u' = 4x^3, u'' = 12x^2, u''' = 24x, u^{(4)} = 24$ (and $u^{(n)} = 0$ for $n > 4$)	M1	1.1b
$v' = 2 \cos(2x), v'' = -4 \sin(2x), v''' = -8 \cos(2x), v^{(4)} = 16 \sin(2x),$ $v^{(5)} = 32 \cos(2x), v^{(6)} = -64 \sin(2x), v^{(7)} = -128 \cos(2x),$ $v^{(8)} = 256 \sin(2x),$	M1 A1 A1	3.1a 1.1b 1.1b
Thus $f^{(8)}(x) = x^4 \times 256 \sin(2x) + 8 \times 4x^3 \times -128 \cos(2x)$ $+ \frac{8 \times 7}{2} \times 12x^2 \times -64 \sin(2x) + \frac{8 \times 7 \times 6}{6} \times 24x \times 32 \cos(2x)$ $+ \frac{8 \times 7 \times 6 \times 5}{24} \times 24 \times 16 \sin(2x)$ $f^{(8)}(x) = x^4 \times 256 \sin(2x) + 8 \times 4x^3 \times -128 \cos(2x)$	M1	2.1

$+28 \times 12x^2 \times -64 \sin(2x) + 56 \times 24x \times 32 \cos(2x)$ $+70 \times 24 \times 16 \sin(2x)$		
$f^{(8)}(\pi) = 0 - 4096\pi^3 - 0 + 1344 \times 2^5\pi + 0$ $(= -4096\pi^3 + 43008\pi)$	M1	1.1b
Coefficient is $\frac{f^{(8)}(\pi)}{8!} = \frac{1344 \times 2^5\pi - 4096\pi^3}{8! \text{ or } \{40320\}}$	M1	2.2a
$= \frac{336\pi - 32\pi^3}{315}$ (So $a = 336$ and $b = -32$)	A1	2.1
	(8)	
	(8 marks)	

Notes

M1: Establishes the non-disappearing derivatives of x^4 . Allow slips in coefficients, but powers must decrease.

M1: Identifies the relevant derivatives for $\sin(2x)$, up to the 8th derivative or establishes the correct pattern. Look for alternating between sin and cos. Condone use of x .

A1: Correct sizes for the coefficients, allow sign errors for this mark (may be due to incorrect signs when differentiating sin and cos) Must have angle $2x$.

A1: All derivatives correctly established. (Note the sin terms may be omitted if the student has made clear they will disappear, but if present they must be correct).

M1: Applies Leibnitz's theorem to get the 8th derivative with their expressions. Binomial coefficients must be present.

M1: Evaluates their 8th derivative at π

M1: Uses Taylor series – divides their value for $f^{(8)}(\pi)$ by 8!

A1: Simplifies to the correct answer.

Note: If do not use Leibnitz's theorem then maximum M0 M0 A0 A0 M0 M1 M1 A0