

QQQ – Further Pure 1 – Chapter 7 – Methods in Calculus (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. Use L'Hospital's rule to show that

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = 0$$

(6)

(Total 6 marks)

Edexcel A2 June 2024 Q3

2.

(a) Show that the substitution $t = \tan\left(\frac{x}{2}\right)$ transforms the integral

$$\int \frac{1}{2 \sin x - \cos x + 5} dx$$

into the integral

$$\int \frac{1}{3t^2 + 2t + 2} dt$$

(4)

(b) Hence determine

$$\int \frac{1}{2 \sin x - \cos x + 5} dx$$

(4)

(Total 8 marks)

Edexcel A2 June 2023 Q5

3.

$$y = e^{3x} \sin x$$

(a) Use Leibnitz's theorem to show that

$$\frac{d^4 y}{dx^4} = 28e^{3x} \sin x + 96e^{3x} \cos x$$

(6)

(b) Hence express $\frac{d^4 y}{dx^4}$ in the form

$$Re^{3x} \sin(x + \alpha)$$

where R and α are constants to be determined, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ **(3)**

(Total 9 marks)

Solutions

1.

Scheme	Marks	AO
$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$	M1	3.1a
$\frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(x \sin x)} = \frac{1 \pm \cos x}{\sin x \pm x \cos x}$	dM1	1.1b
$\frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(x \sin x)} = \frac{1 - \cos x}{\sin x + x \cos x}$	A1	1.1b
$\frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x + x \cos x)} = \frac{\pm \sin x}{\pm \cos x \pm \cos x \pm x \sin x}$	ddM1	3.1a
$\frac{\left(\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \right) \lim_{x \rightarrow 0} \left(\frac{\sin x}{2 \cos x - x \sin x} \right) =}{\frac{\sin(0)}{2 \cos(0) - (0) \sin(0)}}$	M1	1.2
$= 0 *$	A1*	2.1
	(6)	
	(6 marks)	

Notes

M1: Complete method to write the function as a quotient.

dM1: Attempts differentiation of both numerator and denominator, including use of product rule with the denominator. Either numerator or denominator of the correct form. May be done separately.

A1: Fully correct differentiation of both numerator and denominator (may be separate).

ddM1: Dependent on previous method mark. Recognises the need to differentiate again and carries out the differentiation to complete the method.

M1: Clear demonstration of L'Hospital's Rule being used to attain a limit, e.g. clear statement using limits reaching an expression that can be evaluated. Both numerator and denominator must be used appropriately. The substitution may be implied by 0/2 being reached from a suitable expression (oe if errors) but must have achieved a non-zero denominator. Not dependent, so may be scored if genuine errors in the first derivative lead to an expression that has a non-zero denominator. They may have only differentiated once.

A1*: Fully correct solution. **Must see clear use of a substitution** of $x = 0$ into their derivatives. Accept as minimum e.g. $\frac{0}{2-0}$ with each term seen evaluated or equivalent working shown. Needs to be a correct line showing substitution before reaching the printed answer with use of some limit notation. All aspects of the proof should be clear for this mark to be awarded and no errors seen.

NB Proceeding to fourth derivatives before evaluating the limit is a correct approach, and may score the final M once a limit is reached, and final A if all aspects are correct.

2.

Mark scheme (a)

Scheme	Marks	AO
$\frac{dt}{dx} = \frac{1+t^2}{2} \text{ or } \frac{dx}{dt} = \frac{2}{1+t^2} \text{ or } dx = \frac{2 dt}{1+t^2} \text{ or}$ $dt = \frac{1+t^2}{2} dx \text{ oe}$	B1	1.1b
$2 \sin x - \cos x + 5 = 2 \left(\frac{2t}{1+t^2} \right) - \left(\frac{1-t^2}{1+t^2} \right) + 5$	M1	1.1a
$\int \frac{1}{2 \sin x - \cos x + 5} dx =$ $\int \frac{1}{2 \left(\frac{2t}{1+t^2} \right) - \left(\frac{1-t^2}{1+t^2} \right) + 5} \times \frac{2 dt}{1+t^2}$	M1	2.1
$= \int \frac{2}{4t - 1 + t^2 + 5 + 5t^2} \times dt = \int \frac{1}{3t^2 + 2t + 2} dt *$	A1*	2.1
	(4)	

Notes

B1: Correct equation in terms of dx , dt and t – can be implied if seen as part of their substitution.

M1: Express $2 \sin x - \cos x + 5$ in terms of t using half angle formulae with at least one correct. Allow if e.g. there are slips in sign when substituting or the $+5$ is missed.

M1: Makes a **complete** substitution to obtain an integral in terms of t only. Allow slips with the substitution of “ dx ” but must be $dx = f(t) dt$ where $f(t) \neq 1$. This mark is also available if the candidate makes errors when attempting to simplify $2 \left(\frac{2t}{1+t^2} \right) - \left(\frac{1-t^2}{1+t^2} \right)$ before attempting the substitution. Condone omission of the integral and dt if the intent is clear.

A1*: Obtains the printed answer (including dt) with at least one intermediate line or aside working showing elimination of the fractions and no errors seen.

Mark scheme (b)

Scheme	Marks	AO
$\int \frac{1}{3t^2 + 2t + 2} dt = \frac{1}{3} \int \frac{1}{t^2 + \frac{2}{3}t + \frac{2}{3}} dt =$ $\frac{1}{3} \int \frac{1}{\left(t + \frac{1}{3}\right)^2 + \dots} dt$ or $\int \frac{1}{\left(\sqrt{3}t + \frac{1}{\sqrt{3}}\right)^2 + \dots} dt$	M1	3.1a
$\frac{1}{3} \int \frac{1}{\left(t + \frac{1}{3}\right)^2 + \frac{5}{9}} dt \text{ or } \int \frac{1}{\left(\sqrt{3}t + \frac{\sqrt{3}}{3}\right)^2 + \frac{5}{3}} dt \text{ (oe)}$	A1	1.1b
$= \frac{1}{3} \times \frac{1}{\frac{\sqrt{5}}{3}} \tan^{-1} \left(\frac{t + \frac{1}{3}}{\frac{\sqrt{5}}{3}} \right) (+c) = \frac{1}{\sqrt{5}} \tan^{-1}(f(x)) (+c)$ (oe)	M1	3.1a
$= \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{3 \tan \left(\frac{x}{2}\right) + 1}{\sqrt{5}} \right) (+c) \text{ (oe)}$	A1	1.1b
	(4)	
	(8 marks)	

Notes

M1: Adopts the correct strategy of completing the square in order to attempt the integration. Must achieve one of the forms shown in scheme. May be seen separately but must be applied to the integral, not just as an attempt to solve the quadratic.

A1: Correct integral with completed square integrand.

M1: For recognising the arctan form for the integration (look for e.g $k \arctan \left(\frac{t + \frac{1}{3}}{b} \right)$) and makes further progress by undoing the substitution to obtain an answer in terms of x

A1: Correct answer. Accept equivalent forms with simplified (and no nested) fractions. No need for c .

3.

Mark scheme (a)

Scheme	Marks	AO
$y = e^{3x} \sin x \quad u = e^{3x} \quad v = \sin x$		
$u' = 3e^{3x}, u'' = 9e^{3x}, u''' = 27e^{3x}, u^{(4)} = 81e^{3x}$	M1 A1	1.1b 1.1b
$v' = \cos x, v'' = -\sin x, v''' = -\cos x, v^{(4)} = \sin x$	M1 A1	1.1b 1.1b
Thus $\frac{d^4 y}{dx^4} = (e^{3x} \times \sin x) + (4 \times 3e^{3x} \times -\cos x) + (6 \times 9e^{3x} \times -\sin x) + (4 \times 27e^{3x} \times \cos x) + (81e^{3x} \times \sin x)$	M1	2.1
$\frac{d^4 y}{dx^4} = 28e^{3x} \sin x + 96e^{3x} \cos x *$	A1*	1.1b
	(6)	

Notes

M1: Finds the correct form of the first four derivatives of e^{3x} . Must all be of the form αe^{3x} .

SC: M1 may also be awarded for *at least* four relevant derivatives found if they do not find all four for the e^{3x} or the $\sin x$.

A1: Correct derivatives. Allow if the general form is identified rather than individual ones given.

SC allow for at least four correct derivatives if not all are found.

M1: Finds the correct form of the first four derivatives of $\sin x$, condone sign slips only.

A1: Correct derivatives.

M1: Applies Leibnitz's theorem to get the 4th derivative with their expressions. Binomial coefficients must be present and correct numerical expressions (not notational forms).

A1*: Correct simplified 4th derivative from fully correct working, including the $\frac{d^4 y}{dx^4} =$ seen at some stage.

Note: If do not use Leibnitz's theorem then up to the first 4 marks can be awarded for using the product rule to differentiate. Accept correct forms for the derivatives.

M1: $\frac{dy}{dx} = \pm e^{3x} \cos x + Ae^{3x} \sin x$ and $\frac{d^2 y}{dx^2} = Ce^{3x} \cos x + De^{3x} \sin x$

A1: $\frac{dy}{dx} = e^{3x} \cos x + 3e^{3x} \sin x$ and $\frac{d^2 y}{dx^2} = 6e^{3x} \cos x + 8e^{3x} \sin x$

dM1: Forms correct next two. **A1:** $\frac{d^3y}{dx^3} = 26e^{3x} \cos x + 18e^{3x} \sin x$ and

$$\frac{d^4y}{dx^4} = 28e^{3x} \sin x + 96e^{3x} \cos x$$

Mark scheme (b)

Scheme	Marks	AO
$R (= \sqrt{28^2 + 96^2}) = 100$	B1	1.1b
$\tan \alpha = \pm \frac{96}{28} \Rightarrow \alpha = \dots (\alpha = 1.287 \dots)$	M1	1.1b
$\left(\frac{d^4y}{dx^4} = \right) 100e^{3x} \sin(x + 1.29)$	A1	2.1
	(3)	
	(9 marks)	

Notes

(b) Note this appears on open as MAA but is being marked as BMA

B1cao: Deduces the correct value of R , simplified, $R = 100$

M1: Attempts the value of α via $\tan \alpha = \pm \frac{B}{A}$ or $\tan \alpha = \pm \frac{A}{B}$ or $\sin \alpha = \pm \frac{B}{R}$ or $\cos \alpha = \pm \frac{A}{R}$

Note that $\alpha = \text{awrt } 1.29$ or $\alpha = \text{awrt } 74^\circ$ imply this mark.

A1: Correct answer, the $\frac{d^4y}{dx^4}$ may be missing but must be given as the correct expression with $\alpha = \text{awrt } 1.29$ not just separate values for R and α