

QQQ – Further Pure 1 – Chapter 6 – Taylor Series (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1.

$$\frac{d^2y}{dx^2} + 4y - \sin x = 0$$

Given that $y = \frac{1}{2}$ and $\frac{dy}{dx} = \frac{1}{8}$ at $x = 0$,

find a series expansion for y in terms of x , up to and including the term in x^3 . (5)

(Total 5 marks)

Edexcel FP2 June 2013 Q3

2.

$$y = \sqrt{(15 + e^{2x})}$$

(a) Show that

$$\frac{d^2y}{dx^2} = \frac{e^{2x}(e^{2x} + 30)}{(15 + e^{2x})^{\frac{3}{2}}}$$

(3)

(b) Hence determine the Maclaurin series expansion for y , in ascending powers of x , up to and including the term in x^2 , giving each term in simplest form. (2)

(c) Use the series expansion for $\sin x$ in ascending powers of x to determine, in simplest form, the first 2 non-zero terms in ascending powers of x of the series for $\sin 3x$. (1)

(d) Use the answers to parts (b) and (c) to show that

$$\lim_{x \rightarrow 0} \frac{\sqrt{(15 + e^{2x})} - 4}{\sin 3x} = \frac{1}{12}$$

(2)

(Total 8 marks)

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3.

$$y = \sec^2 x$$

(a) Show that $\frac{d^2y}{dx^2} = 6 \sec^4 x - 4 \sec^2 x$. **(4)**

(b) Find a Taylor series expansion of $\sec^2 x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$, up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$. **(6)**

(Total 10 marks)

Solutions

1.

Scheme	Marks
$(x = 0) \quad \frac{d^2y}{dx^2} = \sin 0 - 4 \times \frac{1}{2} = -2$	B1
$\frac{d^3y}{dx^3} + 4\frac{dy}{dx} - \cos x (= 0)$	M1
$(x = 0) \quad \frac{d^3y}{dx^3} = \cos 0 - 4 \times \frac{1}{8} = \frac{1}{2}$	A1
$(y =) y_0 + x \left(\frac{dy}{dx} \right)_0 + \frac{x^2}{2!} \left(\frac{d^2y}{dx^2} \right)_0 + \frac{x^3}{3!} \left(\frac{d^3y}{dx^3} \right)_0 + \dots$	M1 (2! or 2 and 3! or 6)
$(y =) \frac{1}{2} + x \times \frac{1}{8} + \frac{x^2}{2} \times (-2) + \frac{x^3}{6} \times \frac{1}{2}$	
$y = \frac{1}{2} + \frac{x}{8} - x^2 + \frac{x^3}{12}$	A1 cao
	(5 marks)

Notes

B1 for $\left(\frac{d^2y}{dx^2} \right)_0 = -2$ wherever seen

M1 for attempting the differentiation of the given equation. To obtain $\frac{d^3y}{dx^3} \pm k \frac{dy}{dx} \pm \cos x (= 0)$ oe

A1 for substituting $x = 0$ to obtain $\left(\frac{d^3y}{dx^3} \right)_0 = \frac{1}{2}$

M1 for using the expansion $[y = f(x)] = f(0) + xf'(0) + \frac{x^2}{2(!)}f''(0) + \frac{x^3}{3!}f'''(0)$ with their values for $\frac{d^3y}{dx^3}$ and $\frac{d^2y}{dx^2}$. Factorial can be omitted in the x^2 term but must be shown explicitly in the x^3 term or implied by further working eg using 6.

A1cao for $y = \frac{1}{2} + \frac{x}{8} - x^2 + \frac{x^3}{12}$ (Ignore any higher powers included) Exact decimals allowed. **Must** include $y = \dots$

Alternative

Scheme	Marks
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$y = \frac{1}{2} + \frac{x}{8} + ax^2 + bx^3 + \dots$	B1
$y'' = 2a + 6bx + \dots$	M1 Diff twice
$2a + 6bx + \dots = \sin x - 4\left(\frac{1}{2} + \frac{x}{8} + ax^2 + bx^3 \dots\right)$	A1 Correct differentiation and equation used
$2a + 2 = 0 \quad a = -1$	M1
$6b + \frac{1}{2} = 1 \quad b = \frac{1}{12}$	
$y = \frac{1}{2} + \frac{x}{8} - x^2 + \frac{x^3}{12}$	Alcao

(corrected from the printed mark scheme: the printed alternative omits the factor 4 in front of the bracket, which is shown in the notes below.)

B1 for $y = \frac{1}{2} + \frac{x}{8} + ax^2 + bx^3 + \dots$

M1 for differentiating this twice to get $y'' = 2a + 6bx + \dots$ (may not be completely correct)

A1 for correct differentiation and using the given equation and the expansion of $\sin x$ to get

$2a + 6bx + \dots = \left(x - \frac{x^3}{6} + \dots\right) - 4\left(\frac{1}{2} + \frac{x}{8} + \dots\right)$ (corrected from the printed mark scheme: $\frac{x^3}{3}$ is printed for $\frac{x^3}{6}$)

M1 for equating coefficients to obtain a value for a or b

A1 cao for $y = \frac{1}{2} + \frac{x}{8} - x^2 + \frac{x^3}{12}$ (Ignore any higher powers included)

2.

Mark scheme (a)

Scheme	Marks	AO
$y = (15 + e^{2x})^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (15 + e^{2x})^{-\frac{1}{2}} \times 2e^{2x}$ or $y^2 = 15 + e^{2x} \Rightarrow 2y \frac{dy}{dx} = 2e^{2x}$	B1	1.1b

$\frac{d^2y}{dx^2} = \frac{(15 + e^{2x})^{\frac{1}{2}} \times Ae^{2x} + B(15 + e^{2x})^{-\frac{1}{2}} \times e^{2x} \times e^{2x}}{15 + e^{2x}}$ $\left\{ \frac{d^2y}{dx^2} = \frac{(15 + e^{2x})^{\frac{1}{2}} \times 2e^{2x} - \frac{1}{2}(15 + e^{2x})^{-\frac{1}{2}} \times 2e^{2x} \times e^{2x}}{15 + e^{2x}} \right\}$ or $\frac{d^2y}{dx^2} = Ae^{2x}(15 + e^{2x})^{-\frac{1}{2}} + Be^{2x} \times e^{2x}(15 + e^{2x})^{-\frac{3}{2}}$ $\left\{ \frac{d^2y}{dx^2} = 2e^{2x}(15 + e^{2x})^{-\frac{1}{2}} - \frac{1}{2} \times 2e^{2x} \times e^{2x}(15 + e^{2x})^{-\frac{3}{2}} \right\}$ Or $A \left(\frac{dy}{dx} \right)^2 + By \frac{d^2y}{dx^2} = Ce^{2x}$ $\left\{ 2 \left(\frac{dy}{dx} \right)^2 + 2y \frac{d^2y}{dx^2} = 4e^{2x} \right\}$	M1	1.1b
$\frac{d^2y}{dx^2} = \frac{(15 + e^{2x}) \times 2e^{2x} - e^{4x}}{(15 + e^{2x})^{\frac{3}{2}}} = \frac{e^{2x}(e^{2x} + 30)}{(15 + e^{2x})^{\frac{3}{2}}} *$	A1*	2.1
	(3)	

Notes

B1: Correct first derivative in any form, may square first and use implicit differentiation

M1: Differentiates again applying the quotient or product rule correctly to achieve the correct form or uses implicit differentiation to achieve a correct form

A1*: Correct proof with no errors. Must show sufficient working.

If uses the product rule they must show a common denominator

If uses the quotient rule there must be evidence to show the multiplication of $(15 + e^{2x})^{\frac{1}{2}}$ top and bottom

If uses implicit differentiation we need to see the substitution for $\frac{dy}{dx}$ and y and rearranging.

Mark scheme (b)

Scheme	Marks	AO
$x = 0 \Rightarrow y = 4, \frac{dy}{dx} = \frac{1}{4}, \frac{d^2y}{dx^2} = \frac{31}{64}$ leading to $\Rightarrow y = 4 + \frac{x}{4} + \frac{31}{64} \frac{x^2}{2!} + \dots = 4 + \frac{x}{4} + \frac{31}{128} x^2$	M1 A1	1.1b 1.1b

	(2)	
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Notes

M1: A full method to obtain the required expansion. I.e. finds the values up to the second derivative and uses a correct formula

A1: Correct simplified expansion with $y =$ or $f(x) =$ seen somewhere in their solution

Mark scheme (c)

Scheme	Marks	AO
$\sin 3x = 3x - \frac{9}{2}x^3 + \dots$	B1	2.2a
	(1)	

Notes

B1: Deduces the correct simplified expansion, ignore any extra terms, whether correct or not

Mark scheme (d)

Scheme	Marks	AO
$\left\{ \lim_{x \rightarrow 0} \right\} \frac{\sqrt{(15 + e^{2x})} - 4}{\sin 3x} = \left\{ \lim_{x \rightarrow 0} \right\} \frac{4 + \frac{x}{4} + \frac{31}{128}x^2 - 4}{3x - \frac{9}{2}x^3} =$ $\left\{ \lim_{x \rightarrow 0} \right\} \frac{\frac{1}{4} + \frac{31}{128}x}{3 - \frac{9}{2}x^2}$	M1	2.1
$\lim_{x \rightarrow 0} \frac{\frac{1}{4} + \frac{31}{128}x}{3 - \frac{9}{2}x^2} \left\{ = \frac{1}{4} \div 3 \right\} = \frac{1}{12} * \text{cso}$	A1*	1.1b
	(2)	
	(8 marks)	

Notes

M1: A rigorous argument using their answers to (b) and (c), showing the cancelling constant term and the division by x in the numerator and denominator to establish the limiting behaviour.

A1*: Correct proof with no errors including correct limiting notation seen in the final stage cso

Note: Using L'Hospital's Rule with the functions $\sqrt{(15 + e^{2x})} - 4$ and $\sin 3x$ is M0A0

Using L'Hospital's Rule with their expansions $\frac{4 + \frac{x}{4} + \frac{31}{128}x^2 - 4}{3x - \frac{9}{2}x^3}$ leading to $\frac{\frac{1}{4} + \frac{31}{64}x}{3 - \frac{27}{2}x^2}$ could score M1A1

3.

Mark scheme (a)

Scheme	Marks
$y = \sec^2 x = (\sec x)^2$	
$\frac{dy}{dx} = 2(\sec x)^1(\sec x \tan x) = 2 \sec^2 x \tan x$ <i>Either $2(\sec x)^1(\sec x \tan x)$ or $2 \sec^2 x \tan x$</i>	B1 aef
Apply product rule: $\left\{ \begin{array}{ll} u = 2 \sec^2 x & v = \tan x \\ \frac{du}{dx} = 4 \sec^2 x \tan x & \frac{dv}{dx} = \sec^2 x \end{array} \right\}$	
$\frac{d^2y}{dx^2} = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$ <i>Two terms added with one of either $A \sec^2 x \tan^2 x$ or $B \sec^4 x$ in the correct form.</i> <i>Correct differentiation</i>	M1 A1
$= 4 \sec^2 x (\sec^2 x - 1) + 2 \sec^4 x$	
Hence, $\frac{d^2y}{dx^2} = 6 \sec^4 x - 4 \sec^2 x$ <i>Applies $\tan^2 x = \sec^2 x - 1$ leading to the correct result.</i>	A1 AG
	(4)

Mark scheme (b)

Scheme	Marks
$y_{\frac{\pi}{4}} = (\sqrt{2})^2 = \underline{2}, \left(\frac{dy}{dx} \right)_{\frac{\pi}{4}} = 2(\sqrt{2})^2(1) = \underline{4}$ <i>Both $y_{\frac{\pi}{4}} = \underline{2}$ and $\left(\frac{dy}{dx} \right)_{\frac{\pi}{4}} = \underline{4}$</i>	B1
$\left(\frac{d^2y}{dx^2} \right)_{\frac{\pi}{4}} = 6(\sqrt{2})^4 - 4(\sqrt{2})^2 = 24 - 8 = 16$ <i>Attempts to substitute $x = \frac{\pi}{4}$ into both terms in the expression for $\frac{d^2y}{dx^2}$.</i>	M1
$\frac{d^3y}{dx^3} = 24 \sec^3 x (\sec x \tan x) - 8 \sec x (\sec x \tan x)$ $= 24 \sec^4 x \tan x - 8 \sec^2 x \tan x$	M1

Two terms differentiated with either $24 \sec^4 x \tan x$ or $-8 \sec^2 x \tan x$ being correct	
$\left(\frac{d^3 y}{dx^3}\right)_{\frac{\pi}{4}} = 24(\sqrt{2})^4(1) - 8(\sqrt{2})^2(1) = 96 - 16 = 80$ $\left(\frac{d^3 y}{dx^3}\right)_{\frac{\pi}{4}} = \underline{80}$	B1
$\sec^2 x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + \frac{16}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{80}{6}\left(x - \frac{\pi}{4}\right)^3 + \dots$ <i>Applies a Taylor expansion with at least 3 out of 4 terms ft correctly.</i> <i>Correct Taylor series expansion.</i>	M1 A1
$\left\{\sec^2 x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + 8\left(x - \frac{\pi}{4}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{4}\right)^3 + \dots\right\}$	
	(6)
	(10 marks)

Notes

(corrected from the printed mark scheme: the line giving 80 is printed with $\left(\frac{d^2 y}{dx^2}\right)_{\frac{\pi}{4}}$ instead of $\left(\frac{d^3 y}{dx^3}\right)_{\frac{\pi}{4}}$, and the two expansion lines are printed as $\sec x \approx \dots$ instead of $\sec^2 x \approx \dots$)