

QQQ – Further Pure 1 – Chapter 5 – The t-formulae (Version 2)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1.

- (a) Given that $t = \tan \frac{x}{2}$ prove that

$$\cos x \equiv \frac{1 - t^2}{1 + t^2}$$

(3)

- (b) Show that the equation

$$3 \tan x - 10 \cos x = 10$$

can be written in the form

$$(t + 2)(at^2 + bt + c) = 0$$

where $t = \tan \frac{x}{2}$ and a, b and c are integers to be determined. **(4)**

- (c) Hence solve, for $-180^\circ < x < 180^\circ$, the equation

$$3 \tan x - 10 \cos x = 10$$

(5)

(Total 12 marks)

2.

- (i) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to prove that

$$\cot x + \tan\left(\frac{x}{2}\right) = \operatorname{cosec} x \quad x \neq n\pi, n \in \mathbb{Z}$$

(2)

(ii)

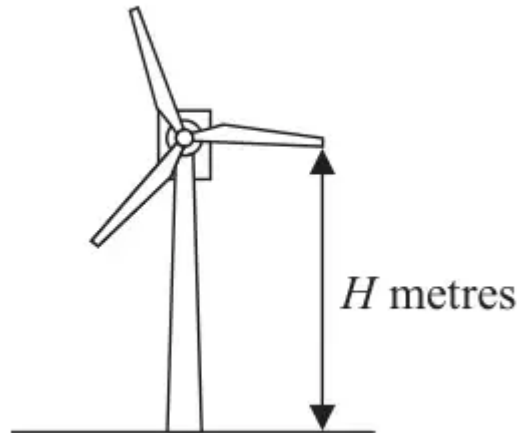


Figure 1

An engineer models the vertical height above the ground of the tip of one blade of a wind turbine, shown in Figure 1. The ground is assumed to be horizontal.

The vertical height of the tip of the blade above the ground, H metres, at time x seconds after the wind turbine has reached its constant operating speed, is modelled by the equation

$$H = 90 - 30 \cos(120x)^\circ - 40 \sin(120x)^\circ \quad (\text{I})$$

- (a) Show that $H = 60$ when $x = 0$ (1)

Using the substitution $t = \tan(60x)^\circ$

- (b) show that equation (I) can be rewritten as

$$H = \frac{120t^2 - 80t + 60}{1 + t^2}$$

(3)

- (c) Hence find, according to the model, the value of x when the tip of the blade is 100 m above the ground for the first time after the wind turbine has reached its constant operating speed. (5)

(Total 11 marks)

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Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$ or $2 \cos^2 \frac{x}{2} - 1$ or $1 - 2 \sin^2 \frac{x}{2}$ or $\tan x = \frac{\tan \frac{x}{2} + \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$	B1	1.2
$\Rightarrow \cos x = \left(\frac{1}{\sqrt{1+t^2}} \right)^2 - \left(\frac{t}{\sqrt{1+t^2}} \right)^2$ or $2 \left(\frac{1}{\sqrt{1+t^2}} \right)^2 - 1$ or $1 - 2 \left(\frac{t}{\sqrt{1+t^2}} \right)^2$ or $\tan x = \frac{2t}{1-t^2} \Rightarrow \cos x = \frac{1-t^2}{\sqrt{(1-t^2)^2 + 4t^2}}$	M1	1.1b
$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2} *$ or $= \frac{2}{1+t^2} - 1 = \frac{2-1-t^2}{1+t^2} = \frac{1-t^2}{1+t^2} *$ or $1 - \frac{2t^2}{1+t^2} = \frac{1+t^2-2t^2}{1+t^2} = \frac{1-t^2}{1+t^2} *$ or $\frac{1-t^2}{\sqrt{(1-t^2)^2 + 4t^2}} = \frac{1-t^2}{\sqrt{t^4 + 2t^2 + 1}} = \frac{1-t^2}{\sqrt{(1+t^2)^2}} = \frac{1-t^2}{1+t^2} *$	A1*	2.1
	(3)	

Notes

B1: Uses any correct appropriate double angle identity for $\cos x$ or possibly $\tan x$.

M1: Substitutes correct expressions for $\cos \frac{x}{2}$ and/or $\sin \frac{x}{2}$ in terms of t

A1*: Completes the proof with sufficient working shown with no errors.

Other alternatives are possible and can be marked in a similar way e.g.

$$\cos x = \frac{\sin x}{\tan x} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\left(\frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} \right)}$$

Scores **B1** for a correct double angle identity for $\cos x$

$$= 2 \left(\frac{1}{\sqrt{1+t^2}} \right) \left(\frac{t}{\sqrt{1+t^2}} \right) \times \frac{1-t^2}{2t}$$

M1 for substituting correct expressions for $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and $\tan \frac{x}{2}$ in terms of t

$$= \frac{2t}{1+t^2} \times \frac{1-t^2}{2t} = \frac{1-t^2}{1+t^2} *$$

A1* for completing the proof with sufficient working shown with no errors.

Allow θ for x for the first 2 marks but must obtain $\cos x = \dots$ for A1*

(a) Special Case – candidates who quote results for $\tan x$ and/or $\sin x$ to verify the result:

e.g.

$$t = \tan \frac{x}{2} \Rightarrow \tan x = \frac{2t}{1-t^2}, \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\sin x}{\tan x} = \frac{2t}{1+t^2} \times \frac{1-t^2}{2t} = \frac{1-t^2}{1+t^2}$$

or e.g.

$$t = \tan \frac{x}{2} \Rightarrow \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{4t^2}{(1+t^2)^2}} = \sqrt{\frac{1+2t^2+t^4-4t^2}{(1+t^2)^2}} = \sqrt{\frac{(1-t^2)^2}{(1+t^2)^2}} = \frac{1-t^2}{1+t^2}$$

Scores SC B1 only

Mark scheme (b)

Scheme	Marks	AO
$3 \tan x - 10 \cos x = 10 \Rightarrow 3 \times \frac{2t}{1-t^2} - 10 \times \frac{1-t^2}{1+t^2} = 10$	B1	1.1a
$\Rightarrow 6t(1+t^2) - 10(1-t^2)^2 = 10(1-t^2)(1+t^2)$ $\Rightarrow 6t + 6t^3 - 10 + 20t^2 - 10t^4 = 10 - 10t^4$ $\Rightarrow 6t^3 + 20t^2 + 6t - 20 = 0$ or e.g. $3t^3 + 10t^2 + 3t - 10 = 0$	M1	2.1
$\Rightarrow (t+2)(\text{"6"}t^2 + \dots t + \dots) = 0$ or e.g. $\Rightarrow (t+2)(\text{"3"}t^2 + \dots t + \dots) = 0$	M1	1.1b

$\Rightarrow (t + 2)(6t^2 + 8t - 10) = 0$ or $(t + 2)(3t^2 + 4t - 5) = 0$	A1	2.2a
	(4)	

Notes

B1: Applies part (a) and the half angle formula for tan to give a correct **equation** in t .

M1: Attempts to multiply by $1 - t^2$ and $1 + t^2$, expands and simplifies to a cubic in t .

M1: Attempts to take a factor $(t + 2)$ out of their cubic.

May be done by inspection or long division but must obtain the correct coefficient for t^2 and reach a 3 term quadratic expression.

A1: Correct equation achieved as shown. Accept with any integer multiple of the quadratic.

Must see the equation written down not just values for a , b and c .

Mark scheme (c)

Scheme	Marks	AO
$t + 2 = 0 \Rightarrow x = 2 \times \arctan(-2) = \dots$	M1	1.1b
$x = -126.86 \dots^\circ$ (allow awrt -127°)	A1	2.2a
$3t^2 + 4t - 5 = 0 \Rightarrow t = \frac{-4 \pm \sqrt{16 - 4 \times 3 \times -5}}{6} =$ $\frac{-2 \pm \sqrt{19}}{3}$ $(= 0.786 \dots, -2.11 \dots)$ $\Rightarrow x = 2 \times \arctan(\dots) = \dots$	M1	3.1a
$x = (\text{awrt}) - 129^\circ, 76.4^\circ \quad (-129.486 \dots, 76.355 \dots)$	A1 (one) A1 (both)	1.1b 1.1b
	(5)	
	(12 marks)	

Notes

M1: Attempts to solve the $t + 2 = 0$ equation, look for an attempt at $\arctan(\pm 2)$ and an attempt to double. Allow in radians for the M mark.

A1: awrt -127°

M1: Solves their quadratic and attempts arctan and doubles.

The usual rules apply for solving the quadratic and may be implied by their values.

A1: Either awrt -129° or awrt 76.4° (degrees symbol not required) or allow for this mark one of these in radians (awrt -2.25 or awrt 1.33)

A1: Both awrt -129° (or better e.g. -129.5°) and awrt 76.4° (degrees symbol not required)
There must be no other values in range for this mark. Values outside the range can be ignored.

2.

Mark scheme (i)

Scheme	Marks	AO
$\text{lhs} = \cot x + \tan\left(\frac{x}{2}\right) = \frac{1-t^2}{2t} + t$	M1	1.1a
$\frac{1-t^2}{2t} + t = \frac{1+t^2}{2t} \left(= \frac{1}{\sin x} \right) = \text{cosec } x *$	A1*	2.1
	(2)	

Notes

M1: Selects the correct expression for $\cot x$ in terms of t and substitutes this and t into the lhs

A1*: Fully correct proof. Allow correct work leading to $\frac{1+t^2}{2t} = \text{cosec } x$

Mark scheme (ii)(a)

Scheme	Marks	AO
$x = 0 \Rightarrow H = 90 - 30 \cos(0) - 40 \sin(0) = 90 - 30 = 60$	B1	1.1b
	(1)	

Notes

B1: Demonstrates that when $x = 0$, $H = 60$

Mark scheme (ii)(b)

Scheme	Marks	AO
$H = 90 - 30 \cos 120x - 40 \sin 120x = 90 - 30 \left(\frac{1-t^2}{1+t^2} \right) - 40 \left(\frac{2t}{1+t^2} \right)$	M1	1.1b
$= \frac{90 + 90t^2 - 30 + 30t^2 - 80t}{1+t^2}$	M1	1.1b

$= \frac{120t^2 - 80t + 60}{1 + t^2} *$	A1*	2.1
	(3)	

Notes

M1: Uses the correct formulae to obtain H in terms of t

M1: Correct method to obtain a common denominator

A1*: Collects terms and simplifies to obtain the printed answer with no errors

Mark scheme (ii)(c)

Scheme	Marks	AO
$\frac{120t^2 - 80t + 60}{1 + t^2} = 100 \Rightarrow 120t^2 - 80t + 60 = 100 + 100t^2$	M1	3.4
$20t^2 - 80t - 40 = 0$	A1	1.1b
$t = \frac{4 \pm \sqrt{16 + 8}}{2} \Rightarrow 60x = \tan^{-1}(2 + \sqrt{6})$ or $60x = \tan^{-1}(2 - \sqrt{6})$	M1	3.4
$60x = \tan^{-1}(2 + \sqrt{6}) = 77.33 \dots \Rightarrow x = \dots$	dM1	3.1b
$x = 1.29$	A1	3.2a
	(5)	
	(11 marks)	

Notes

M1: Uses $H = 100$ with the model and multiplies up to obtain a quadratic equation in t

A1: Correct 3TQ

M1: Solves their 3TQ in t and proceeds to obtain values of $60x$ as suggested by the model

M1: A fully correct strategy to identify the required value of x from the positive root of the quadratic equation in t

A1: awrt 1.29

Attempts in radians can score all but the final mark in (c). (Gives $60x = 1.3 \dots$ etc.)