

QQQ – Further Pure 1 – Chapter 5 – The t-formulae (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1.

(a) Write down the t -formula for $\sin x$. **(1)**

(b) Use the answer to part (a)

(i) to find the exact value of $\sin x$ when

$$\tan\left(\frac{x}{2}\right) = \sqrt{2}$$

(ii) to show that

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

(4)

(c) Use the t -formulae to solve for $0 < \theta \leq 360^\circ$

$$7 \sin \theta + 9 \cos \theta + 3 = 0$$

giving your answers to one decimal place. **(4)**

(Total 9 marks)

2.

- (a) Use $t = \tan \frac{\theta}{2}$ to show that, where both sides are defined

$$\frac{29 - 21 \sec \theta}{20 - 21 \tan \theta} \equiv \frac{5t + 2}{2t + 5}$$

(4)

- (b) Hence, again using $t = \tan \frac{\theta}{2}$, prove that, where both sides are defined

$$\frac{20 + 21 \tan \theta}{29 + 21 \sec \theta} \equiv \frac{29 - 21 \sec \theta}{20 - 21 \tan \theta}$$

(3)

(Total 7 marks)

Edexcel AS June 2022 Q3

3. During 2029, the number of hours of daylight per day in London, H , is modelled by the equation

$$H = 0.3 \sin \left(\frac{x}{60} \right) - 4 \cos \left(\frac{x}{60} \right) + 11.5 \quad 0 \leq x < 365$$

where x is the number of days after 1st January 2029 and the angle is in radians.

- (a) Show that, according to the model, the number of hours of daylight in London on the 31st January 2029 will be 8.13 to 3 significant figures. (1)
- (b) Use the substitution $t = \tan \left(\frac{x}{120} \right)$ to show that H can be written as

$$H = \frac{at^2 + bt + c}{1 + t^2}$$

where a , b and c are constants to be determined. (2)

- (c) Hence determine, according to the model, the date of the first day of 2029 when there will be at least 12 hours of daylight in London. (4)

(Total 7 marks)

Edexcel A2 June 2022 Q2

Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
$\{\sin x =\} \frac{2t}{1+t^2}$	B1	1.2
	(1)	

Notes

B1: See scheme

Mark scheme (b)

Scheme	Marks	AO
(b)(i) $\left\{ \tan \left(\frac{x}{2} \right) = \sqrt{2} \Rightarrow t = \sqrt{2} \Rightarrow \right\} \sin x = \frac{2(\sqrt{2})}{1 + (\sqrt{2})^2}$ or $\frac{2(\sqrt{2})}{1 + 2}$	M1	1.1b
$\sin x = \frac{2}{3}\sqrt{2}$ or $\frac{1}{3}\sqrt{8}$ or $\sqrt{\frac{8}{9}}$	A1	1.1b
	(2)	
(b)(ii) Way 1 $\left\{ \cos x \equiv \frac{\sin x}{\tan x} \Rightarrow \right\} \cos x = \frac{\frac{2t}{1+t^2}}{\frac{2t}{1-t^2}} ; = \frac{1-t^2}{1+t^2} * \text{CSO}$	M1; A1*	1.1b 2.1
	(2)	

Notes

(b)(i)

M1: Complete substitution of $t = \sqrt{2}$ into their expression from part (a)

A1: Correct exact answer. See scheme.

Note: Give M0 A0 for writing down the correct exact answer without any evidence of substituting $t = \sqrt{2}$

into $\sin x = \frac{2t}{1+t^2}$

Note: For reference, $\sin x = \frac{2}{3}\sqrt{2} = 0.9428 \dots$

(b)(ii) Way 1, Way 2 and Way 3

M1: Uses a correct trigonometric identity (or correct trigonometric identities) to find a correct expression which connects only $\cos x$ (or $\cos^2 x$) and t

A1*: Correct proof

(b)(ii) Way 4

M1: Uses $\sin x = \frac{o}{h}$ and a correct Pythagoras method to express the adjacent edge of a triangle in terms of t .

A1*: Correct proof

(b)(ii) Way 2

Scheme	Marks	A0
$\left\{ \tan x \equiv \frac{\sin x}{\cos x} \Rightarrow \right\} \frac{2t}{1-t^2} = \frac{\frac{2t}{1+t^2}}{\cos x} ; \Rightarrow \cos x = \frac{1-t^2}{1+t^2} *$ CSO	M1; A1*	1.1b 2.1
	(2)	

(b)(ii) Way 3

Scheme	Marks	A0
$\left\{ \sin^2 x + \cos^2 x \equiv 1 \Rightarrow \right\} \left(\frac{2t}{1+t^2} \right)^2 + \cos^2 x = 1$	M1	1.1b
$\cos^2 x = 1 - \left(\frac{2t}{1+t^2} \right)^2 = \frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2} = \frac{1-2t^2+t^4}{(1+t^2)^2} =$ $\frac{(1-t^2)^2}{(1+t^2)^2}$ $\Rightarrow \cos x = \frac{1-t^2}{1+t^2} * \text{CSO}$	A1	2.1
	(2)	

(b)(ii) Way 4

Scheme	Marks	A0
$\{o^2 + a^2 = h^2 \Rightarrow\} (2t)^2 + a^2 = (1+t^2)^2$	M1	1.1b

$a^2 = (1 + t^2)^2 - (2t)^2 = 1 - 2t^2 + t^4 = (1 - t^2)^2$ $a = 1 - t^2 \Rightarrow \cos x = \frac{1 - t^2}{1 + t^2} * \text{CSO}$	A1	2.1
	(2)	

Mark scheme (c)

Scheme	Marks	AO
$\{7 \sin \theta + 9 \cos \theta + 3 =\} 7 \left(\frac{2t}{1 + t^2} \right) + 9 \left(\frac{1 - t^2}{1 + t^2} \right) + 3$	M1	1.1b
$7 \left(\frac{2t}{1 + t^2} \right) + 9 \left(\frac{1 - t^2}{1 + t^2} \right) + 3 = 0 \Rightarrow 14t + 9 - 9t^2 + 3 + 3t^2 = 0$ $\Rightarrow 6t^2 - 14t - 12 = 0 \Rightarrow 3t^2 - 7t - 6 = 0 \Rightarrow (t - 3)(3t + 2) = 0 \Rightarrow t = \dots$	M1	1.1b
Either $\left\{ t = 3 \Rightarrow \frac{\theta}{2} = \arctan(3) \Rightarrow \right\} \theta = 2 \arctan(3)$ or $\left\{ t = -\frac{2}{3} \Rightarrow \frac{\theta}{2} = 180^\circ + \arctan\left(-\frac{2}{3}\right) \Rightarrow \right\} \theta =$ $2 \left(180^\circ + \arctan\left(-\frac{2}{3}\right) \right)$	M1	1.1b
$\frac{\theta}{2} = \{71.5650\dots, 146.3099\dots\} \Rightarrow \theta =$ $\{143.1301\dots, 292.6198\dots\}$		
$\theta = 143.1^\circ, 292.6^\circ$ (1dp)	A1	1.1b
	(4)	
	(9 marks)	

Notes

M1: Uses at least one of $\sin \theta = \frac{2t}{1 + t^2}$ or $\cos \theta = \frac{1 - t^2}{1 + t^2}$ to express $7 \sin \theta + 9 \cos \theta + 3$ in terms of t only

M1: Uses both correct formula $\sin \theta = \frac{2t}{1 + t^2}$ and $\cos \theta = \frac{1 - t^2}{1 + t^2}$ in $7 \sin \theta + 9 \cos \theta + 3 = 0$, multiplies both sides by $1 + t^2$, forms a 3TQ and uses a correct method (e.g. using the quadratic formula, completing the square or a calculator approach) for solving their 3TQ to give $t = \dots$

M1: Uses both correct formula $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ in $7 \sin \theta + 9 \cos \theta + 3 = 0$, adopts a correct *applied* strategy to find at least one value of θ within the range $0 < \theta \leq 360^\circ$ (or in radians $0 < \theta \leq 2\pi$) such that either

- $\theta = 2 \arctan(\text{their found } t)$, where their found $t > 0$
- $\theta = 2(180^\circ + \arctan(\text{their found } t))$, where their found $t < 0$
- $\theta = 2(180^\circ - \arctan |\text{their found } t|)$, where their found $t < 0$

A1: Correct answer only of $\theta = 143.1^\circ, 292.6^\circ$

Note: Give A0 for extra solutions given within the range $0 < \theta \leq 360^\circ$

Note: Ignore extra solutions outside the range $0 < \theta \leq 360^\circ$ for the A mark

Note: Give 3rd M0 for $\frac{\theta}{2} = \{71.565 \dots, 146.309 \dots\}$ without attempting to find θ

Note: Give 3rd M0 for $\frac{\theta}{2} = \{71.565 \dots, 146.309 \dots\} \Rightarrow \theta = \{35.782 \dots, 73.154 \dots\}$

Note: In degrees, $\frac{\theta}{2} = \{71.565 \dots, 251.565 \dots, -33.690 \dots, 146.309 \dots\}$

Note: Working in radians gives $\frac{\theta}{2} = \{1.249 \dots, 2.553 \dots\} \Rightarrow \theta = \{2.498 \dots, 5.107 \dots\}$

2.

Mark scheme (a)

Scheme	Marks	AO
Sight of $\sec \theta = \frac{1+t^2}{1-t^2}$ and $\tan \theta = \frac{2t}{1-t^2}$ at least once each.	B1	1.2
$\frac{29 - 21 \sec \theta}{20 - 21 \tan \theta} = \frac{29 - 21 \left(\frac{1+t^2}{1-t^2} \right)}{20 - 21 \left(\frac{2t}{1-t^2} \right)}$	M1	1.1b
$= \frac{29(1-t^2) - 21(1+t^2)}{20(1-t^2) - 21(2t)} = \frac{8 - 50t^2}{-20t^2 - 42t + 20}$ or $= \frac{\left(\frac{29(1-t^2) - 21(1+t^2)}{(1-t^2)} \right)}{\left(\frac{20(1-t^2) - 21(2t)}{(1-t^2)} \right)} = \frac{8 - 50t^2}{-20t^2 - 42t + 20}$	M1	2.1

$= \frac{-2(5t-2)(5t+2)}{-2(5t-2)(2t+5)} = \frac{5t+2}{2t+5} * \text{CSO}$ $= \frac{2(2+5t)(2-5t)}{2(2-5t)(5+2t)} = \frac{5t+2}{2t+5} * \text{CSO}$	A1*	1.1b
	(4)	

Notes

B1: Uses the correct identities at least once each. May be seen in (a) or (b)

M1: Substitutes their identities into the equation, they need not be correct

M1: Multiplies numerator and denominator by $1 - t^2$ and simplifies to a quadratic with all terms collected

Special case: If they make an error with the identities award M1 if they use the correct method to write the numerator and denominator as a single fraction and then divides to achieve an expression of the form $\frac{a}{b}$

A1*: Cancels factors to achieve the given expression, with no errors seen

Condone $\frac{50t^2 - 8}{20t^2 + 42t - 20} = \frac{(5t+2)(5t-2)}{(5t-2)(2t+5)} = \frac{5t+2}{2t+5}$ and

$\frac{-8 + 50t^2}{-20t^2 - 42t + 20} = \frac{(5t+2)(5t-2)}{(5t-2)(2t+5)} = \frac{5t+2}{2t+5}$

Mark scheme (b)

Scheme	Marks	AO
$\frac{20 + 21 \tan \theta}{29 + 21 \sec \theta} = \frac{20 + 21 \left(\frac{2t}{1-t^2} \right)}{29 + 21 \left(\frac{1+t^2}{1-t^2} \right)}$ $= \frac{20(1-t^2) + 21(2t)}{29(1-t^2) + 21(1+t^2)}$	M1	2.1
$\frac{20 + 42t - 20t^2}{50 - 8t^2} = \frac{2(5-2t)(5t+2)}{2(5-2t)(5+2t)} = \dots$ $\frac{20 + 42t - 20t^2}{50 - 8t^2} = \frac{-2(2t-5)(5t+2)}{-2(2t+5)(2t-5)} = \dots$	M1	1.1b
Achieves form correct working = $\frac{5t+2}{2t+5} *$ Then concludes hence the result is true or	A1*	2.2a

$= \text{RHS}$ or $= \frac{5t + 2}{2t + 5} = \frac{29 - 21 \sec \theta}{20 - 21 \tan \theta}$ or $\frac{20 + 21 \tan \theta}{29 + 21 \sec \theta} = \frac{29 - 21 \sec \theta}{20 - 21 \tan \theta}$ LHS = RHS		
	(3)	
	(7 marks)	

Notes

M1: Substitutes into LHS of equation and multiplies numerator and denominator by $1 - t^2$ to work towards the other side.

M1: Simplifies to a quadratic with all terms collected and factorises then cancels terms in the LHS in an attempt to try and match expressions.

A1*: Achieves correct expressions for both sides and gives a conclusion deducing that the result is true.

M1A0 for $\frac{20t^2 - 42t - 20}{8t^2 - 50} = \frac{(5t + 2)(2t - 5)}{(2t - 5)(2t + 5)} = \frac{5t + 2}{2t + 5}$ and

$$\frac{-20t^2 - 42t + 20}{-8 + 50t^2} = \frac{(5t + 2)(2t - 5)}{(2t - 5)(2t + 5)} = \frac{5t + 2}{2t + 5}$$

3.

Mark scheme (a)

Scheme	Marks	AO
$(H =) 0.3 \sin \left(\frac{30}{60} \right) - 4 \cos \left(\frac{30}{60} \right) + 11.5 = 8.13 \text{ {hours}}^*$	B1*	3.4
	(1)	

Notes

B1*: Uses $x = 30$ to show that $H = 8.13$. Accept $x = 30$ seen substituted followed by 8.13, or $x = 30$ identified followed by 8.133... before rounding to 8.13.

Mark scheme (b)

Scheme	Marks	AO
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Substitutes $\sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2}$ and $\cos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}$ into H $(H =) 0.3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 11.5$	M1	1.1b
$(H =) \frac{0.6t - 4 + 4t^2 + 11.5(1+t^2)}{1+t^2} =$ $\frac{15.5t^2 + 0.6t + 7.5}{1+t^2}$	A1	2.1
	(2)	

Notes

M1: Uses the correct t -formulae $\sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2}$ and $\cos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}$, attempts to substitute into H .

A1: Fully correct method, expresses as a single fraction with a denominator of $1+t^2$ to achieve

$$H = \frac{15.5t^2 + 0.6t + 7.5}{1+t^2} \text{ (oe with fractions or accept values for } a, b \text{ and } c \text{ stated).}$$

Mark scheme (c)

Scheme	Marks	AO
$H = \frac{15.5t^2 + 0.6t + 7.5}{1+t^2} = 12 \Rightarrow 3.5t^2 + 0.6t - 4.5 = 0$	M1	3.4
$\Rightarrow t = \frac{-0.6 \pm \sqrt{0.6^2 - 4(3.5)(-4.5)}}{7}$ $= \dots (1.051\dots, -1.222\dots) \Rightarrow x =$ $120 \tan^{-1}(\text{"1.051..."}) = \dots (97.254\dots)$	dM1	3.1b
$x = \text{awrt } 97$	A1	1.1b
8 th or 9 th April	A1	3.2a
	(4)	
	(7 marks)	

Notes

M1: Sets $H = 12$ (or any inequality in between) and rearranges to form a quadratic equation for t .

dM1: Dependent on the previous method mark. Solves the quadratic by any means (accept one correct answer for their quadratic if no method shown) and uses this to find a value for x .

A1: Correct value for x = awrt 97 or accept 98 following a correct value for t .

A1: Correct day of the year. Accept 8th or 9th April following awrt 97 from a correct method.

Note: Question says hence, so answers by graphical methods or trial and improvement are not acceptable for full credit. They can score a SC M0dM0A0B1 for achieving a correct date.