

# QQQ – Decision 2 – Chapter 6 – Game Theory (Version 2)

**Total Marks: 22**

(22 = Platinum, 20 = Gold, 18 = Silver, 15 = Bronze)

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1. Anna (A) and Roland (R) play a two-person zero-sum game which is represented by the following pay-off matrix for Anna.

|           | R plays 1 | R plays 2 | R plays 3 |
|-----------|-----------|-----------|-----------|
| A plays 1 | 6         | -2        | -3        |
| A plays 2 | -3        | 1         | 2         |
| A plays 3 | 5         | 4         | -1        |

Formulate the game as a linear programming problem **for player R**. Write the constraints as inequalities. Define your variables clearly.

**(8)**

**(Total 8 marks)**

2. A two-person zero-sum game is represented by the following pay-off matrix for player  $A$ .

|             | $B$ plays X | $B$ plays Y |
|-------------|-------------|-------------|
| $A$ plays Q | 2           | -2          |
| $A$ plays R | -1          | 5           |
| $A$ plays S | 3           | 4           |
| $A$ plays T | 0           | 2           |

- (a) (i) Show that this game is stable.
- (ii) State the value of this game to player  $B$ . (3)

Option S is removed from player  $A$ 's choices and the reduced game, with option S removed, is no longer stable.

- (b) Write down the reduced pay-off matrix for player  $B$ . (1)

Let  $B$  play option X with probability  $p$  and option Y with probability  $1 - p$ .

- (c) Use a graphical method to find the optimal value of  $p$  and hence find the best strategy for player  $B$  in the reduced game. (6)
- (d) (i) Find the value of the reduced game to player  $A$ .
- (ii) State which option player  $A$  should never play in the reduced game.
- (iii) Hence find the best strategy for player  $A$  in the reduced game. (4)

**(Total 14 marks)**

# Solutions

1.

| Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Marks     |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|----|----|----|----------------|----|---|----|----------------|---|----|----|----------------|---|----|---|--|----|----|----|----------------|---|----|---|----------------|---|---|---|----------------|----|---|---|--------|
| <u>Alt 1</u>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |           |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| Game from R's point of view.<br><br><table><tr><td></td><td>A1</td><td>A2</td><td>A3</td></tr><tr><td>R<sub>1</sub></td><td>-6</td><td>3</td><td>-5</td></tr><tr><td>R<sub>2</sub></td><td>2</td><td>-1</td><td>-4</td></tr><tr><td>R<sub>3</sub></td><td>3</td><td>-2</td><td>1</td></tr></table><br><br>Add 7<br><br><table><tr><td></td><td>A1</td><td>A2</td><td>A3</td></tr><tr><td>R<sub>1</sub></td><td>1</td><td>10</td><td>2</td></tr><tr><td>R<sub>2</sub></td><td>9</td><td>6</td><td>3</td></tr><tr><td>R<sub>3</sub></td><td>10</td><td>5</td><td>8</td></tr></table> |           | A1 | A2 | A3 | R <sub>1</sub> | -6 | 3 | -5 | R <sub>2</sub> | 2 | -1 | -4 | R <sub>3</sub> | 3 | -2 | 1 |  | A1 | A2 | A3 | R <sub>1</sub> | 1 | 10 | 2 | R <sub>2</sub> | 9 | 6 | 3 | R <sub>3</sub> | 10 | 5 | 8 | B1, B1 |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | A1        | A2 | A3 |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| R <sub>1</sub>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | -6        | 3  | -5 |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| R <sub>2</sub>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 2         | -1 | -4 |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| R <sub>3</sub>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 3         | -2 | 1  |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | A1        | A2 | A3 |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| R <sub>1</sub>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 1         | 10 | 2  |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| R <sub>2</sub>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 9         | 6  | 3  |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| R <sub>3</sub>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 10        | 5  | 8  |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| Let R play 1 with probability P <sub>1</sub> , 2 with probability P <sub>2</sub> , 3 with probability P <sub>3</sub><br>V = value of the game                                                                                                                                                                                                                                                                                                                                                                                                                                      | B1        |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| Maximise P = V                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | B1        |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| Subject to V – P <sub>1</sub> – 9P <sub>2</sub> – 10P <sub>3</sub> ≤ 0                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | M1 A1ft   |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| V – 10P <sub>1</sub> – 6P <sub>2</sub> – 5P <sub>3</sub> ≤ 0                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | A1ft      |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| V – 2P <sub>1</sub> – 3P <sub>2</sub> – 8P <sub>3</sub> ≤ 0                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | A1ft      |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
| P <sub>1</sub> + P <sub>2</sub> + P <sub>3</sub> ≤ 1 accept =<br>V, P <sub>1</sub> , P <sub>2</sub> , P <sub>3</sub> ≥ 0                                                                                                                                                                                                                                                                                                                                                                                                                                                           | A1        |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | (8 marks) |    |    |    |                |    |   |    |                |   |    |    |                |   |    |   |  |    |    |    |                |   |    |   |                |   |   |   |                |    |   |   |        |

Notes

Alt 2

|                                                                                                                                                                                                                                                                                          |         |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------|-------|-------|-------|-----------|----|---|---|-----------|---|---|---|-----------|---|---|---|--|
| Add 4 to all entries                                                                                                                                                                                                                                                                     | B1      |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| <table><tr><td></td><td><math>R_1</math></td><td><math>R_2</math></td><td><math>R_3</math></td></tr><tr><td><b>A1</b></td><td>10</td><td>2</td><td>1</td></tr><tr><td><b>A2</b></td><td>1</td><td>5</td><td>6</td></tr><tr><td><b>A3</b></td><td>9</td><td>8</td><td>3</td></tr></table> |         | $R_1$ | $R_2$ | $R_3$ | <b>A1</b> | 10 | 2 | 1 | <b>A2</b> | 1 | 5 | 6 | <b>A3</b> | 9 | 8 | 3 |  |
|                                                                                                                                                                                                                                                                                          | $R_1$   | $R_2$ | $R_3$ |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| <b>A1</b>                                                                                                                                                                                                                                                                                | 10      | 2     | 1     |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| <b>A2</b>                                                                                                                                                                                                                                                                                | 1       | 5     | 6     |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| <b>A3</b>                                                                                                                                                                                                                                                                                | 9       | 8     | 3     |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| Let R play 1 with probability $P_1$ , 2 with probability $P_2$ , 3 with probability $P_3$ ; let $V$ = value of game.                                                                                                                                                                     | B1      |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| Let $x_1 = \frac{P_1}{V}$ , $x_2 = \frac{P_2}{V}$ , $x_3 = \frac{P_3}{V}$                                                                                                                                                                                                                | B1      |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| Maximise $P = x_1 + x_2 + x_3$                                                                                                                                                                                                                                                           | B1      |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| Subject to $10x_1 + 2x_2 + x_3 \leq 1$                                                                                                                                                                                                                                                   | M1 A1ft |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| $x_1 + 5x_2 + 6x_3 \leq 1$                                                                                                                                                                                                                                                               | A1ft    |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |
| $9x_1 + 8x_2 + 3x_3 \leq 1$<br>$x_1, x_2, x_3 \geq 0$ accept $P_i \geq 0$                                                                                                                                                                                                                | A1      |       |       |       |           |    |   |   |           |   |   |   |           |   |   |   |  |

The codes listed for Alt 1 add up to 9, but the question total is 8 marks.

## 2.

### Mark scheme (a)

| Scheme                                                                           | Marks      | A0   |
|----------------------------------------------------------------------------------|------------|------|
| <b>(i)</b> Row minima: $-2, -1, 3, 0$ max is 3<br>Column maxima: $3, 5$ min is 3 | M1         | 1.1b |
| Row(maximin) = Col(minimax) which are both 3 therefore game is stable            | A1         | 2.4  |
| <b>(ii)</b> Value of the game to player $B$ is $-3$                              | A1         | 2.2a |
|                                                                                  | <b>(3)</b> |      |

### Notes

**M1:** finding row minimums and column maximums – condone one error

**A1:** row maximin 3 = col minimax 3 (so stable) – dependent on all 6 correct values – as a minimum must see the two 3's explicitly being considered

**A1:** cao  $(-3)$  – ‘lose 3’ is **A0**

### Mark scheme (b)

| Scheme                                                    | Marks      | AO   |
|-----------------------------------------------------------|------------|------|
| $\begin{pmatrix} -2 & 1 & 0 \\ 2 & -5 & -2 \end{pmatrix}$ | B1         | 1.1b |
|                                                           | <b>(1)</b> |      |

### Notes

**B1:** cao (must be the reduced game so with option S removed)

### Mark scheme (c)

| Scheme                                                                                                                                                                                                                                       | Marks               | AO                      |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------|-------------------------|
| <p>If A plays option Q, B's gains are <math>-2p + 2(1 - p) = 2 - 4p</math></p> <p>If A plays option R, B's gains are <math>p + (-5)(1 - p) = -5 + 6p</math></p> <p>If A plays option T, B's gains are <math>(-2)(1 - p) = -2 + 2p</math></p> | <p>M1</p> <p>A1</p> | <p>1.1b</p> <p>1.1b</p> |
|                                                                                                                                                                                                                                              | <p>M1</p> <p>A1</p> | <p>1.1b</p> <p>1.1b</p> |
| $2 - 4p = -5 + 6p \Rightarrow p = 7/10$                                                                                                                                                                                                      | A1                  | 1.1b                    |
| B should play option X with probability 7/10 and option Y with probability 3/10                                                                                                                                                              | A1ft                | 3.2a                    |
|                                                                                                                                                                                                                                              | <b>(6)</b>          |                         |

## Notes

In (c) if the candidate includes option S (and so has four expressions and lines) then this can score the first 3 marks only in this part

**M1:** setting up three expressions in terms of  $p$

**A1:** all three expressions correctly simplified

**M1:** at least two lines correctly drawn for their expressions – if values on at least one vertical axis not given then lines must be in the right position relative to each other

**A1:** completely correct graph with clear indication of the correct intersection points at the ends where  $p = 0$  and  $p = 1$  (so if no scaling on the vertical axis assume that 1 line = 1 unit (unless other suitable scaling is clear), and lines must not extend past  $p < 0$  and/or  $p > 1$ )

**A1:** using the graph (with 3 lines) to obtain the correct probability expressions leading to the correct value of  $p$

**A1ft:** interpret their value of  $p$  in the context of the question – must refer to ‘play’ and the associated probabilities (but do not need to explicitly use the word ‘probability’)

## Mark scheme (d)

| Scheme                                                                                                                                   | Marks      | AO   |
|------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
| (i) $6\left(\frac{7}{10}\right) - 5 = -4/5 \Rightarrow$ value of the game to player $A$ is $4/5$                                         | B1         | 3.1a |
| (ii) Player $A$ should never play option T                                                                                               | B1         | 2.2a |
| (iii) If $A$ plays their option Q with probability $q$ and their option R with probability $1 - q$ then $2q + (-1)(1 - q) = \frac{4}{5}$ | M1         | 3.1b |
| $q = \frac{3}{5} \Rightarrow A$ should play option Q with probability $3/5$ and option R with probability $2/5$ (play S and T never)     | A1         | 3.2a |
|                                                                                                                                          | (4)        |      |
|                                                                                                                                          | (14 marks) |      |

## Notes

(d)(i) **B1:** cao ( $\frac{4}{5}$  or 0.8)

(d)(ii) **B1:** cao (option T)

(d)(iii) **M1:** Setting up a linear equation with their  $V(A)$  from (d)(i) wither either of the two correct expressions  $2q + (-1)(1 - q)$  **or**  $-2q + 5(1 - q)$  **or** setting up the correct equation

$$-2q + 5(1 - q) = 2q + (-1)(1 - q) \text{ (or } 2q - 5(1 - q) = -2q + (1 - q)\text{)}$$

**A1:** interpret the correct value of  $q$  in the context of the question – must refer to ‘play’ and the associated probabilities (but do not need to explicitly use the word ‘probability’)

(corrected from the printed mark scheme: the printed line in (d)(i) reads “ $6 \frac{7}{10} - 5 = -4/5$ ”, with the bracket/multiplication between 6 and  $7/10$  missing)