

QQQ – Decision 2 – Chapter 2 – Allocation Problems (Version 1)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1. Four workers, A, B, C and D, are to be assigned to four tasks, P, Q, R and S. Each worker must be assigned to exactly one task and each task must be done by only one worker. The time, in hours, that each worker takes to complete each task is shown in the table below.

	P	Q	R	S
A	7.5	3.5	8	9.5
B	5	2	7	7.5
C	4	3.5	3.5	8
D	6	5	3.5	4

Reducing rows first, use the Hungarian algorithm to obtain an allocation which minimises the total time. You must explain your method and show the table after each stage. **(5)**

(Total 5 marks)

2. Five workers, A, B, C, D and E, are available to complete four tasks, P, Q, R and S.

Each worker can only be assigned to at most one task, and each task must be done by at most one worker.

Worker B cannot be assigned to task Q and worker E cannot be assigned to task S.

The time, in minutes, that each worker takes to complete each task is shown in the table below.

	P	Q	R	S
A	38	39	37	37
B	39	–	39	40
C	41	44	40	42
D	40	41	39	38
E	36	39	41	–

The Hungarian algorithm is to be used to find the least total time to complete all four tasks.

- (a) Explain how the table should be modified so that the Hungarian algorithm can be applied. (2)
- (b) (i) Use the Hungarian algorithm to obtain an allocation that minimises the total time.
(ii) Explain how you determined if the table was optimal at each stage. (6)
- (c) Calculate the least total time to complete all four tasks. (1)

(Total 9 marks)

3. Four workers, A, B, C and D, are to be assigned to four tasks, P, Q, R and S.

Each task must be assigned to just one worker and each worker can do only one task.

Worker B cannot be assigned to task Q and worker D cannot be assigned to task R.

The amount, in pounds, that each worker would earn when assigned to each task is shown in the table below.

	P	Q	R	S
A	65	72	69	75
B	71	–	68	65
C	70	69	73	77
D	73	70	–	71

The Hungarian algorithm can be used to find the maximum total amount that would be earned by the four workers.

- (a) (i) Explain how to modify the table so that the Hungarian algorithm could be applied.
(ii) Modify the table as described in (a)(i). **(3)**
- (b) Formulate the above situation as a linear programming problem. You must define the decision variables and make the objective function and constraints clear. **(6)**

(Total 9 marks)

Solutions

1.

Scheme	Marks	AO																																																		
Reducing rows <table><tr><td></td><td>P</td><td>Q</td><td>R</td><td>S</td></tr><tr><td>A</td><td>4</td><td>0</td><td>4.5</td><td>6</td></tr><tr><td>B</td><td>3</td><td>0</td><td>5</td><td>5.5</td></tr><tr><td>C</td><td>0.5</td><td>0</td><td>0</td><td>4.5</td></tr><tr><td>D</td><td>2.5</td><td>1.5</td><td>0</td><td>0.5</td></tr></table> and columns <table><tr><td></td><td>P</td><td>Q</td><td>R</td><td>S</td></tr><tr><td>A</td><td>3.5</td><td>0</td><td>4.5</td><td>5.5</td></tr><tr><td>B</td><td>2.5</td><td>0</td><td>5</td><td>5</td></tr><tr><td>C</td><td>0</td><td>0</td><td>0</td><td>4</td></tr><tr><td>D</td><td>2</td><td>1.5</td><td>0</td><td>0</td></tr></table>		P	Q	R	S	A	4	0	4.5	6	B	3	0	5	5.5	C	0.5	0	0	4.5	D	2.5	1.5	0	0.5		P	Q	R	S	A	3.5	0	4.5	5.5	B	2.5	0	5	5	C	0	0	0	4	D	2	1.5	0	0	M1 A1	1.1b 1.1b
	P	Q	R	S																																																
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C	0	0	0	4																																																
D	2	1.5	0	0																																																
augment by 2.5 <table><tr><td></td><td>P</td><td>Q</td><td>R</td><td>S</td></tr><tr><td>A</td><td>1</td><td>0</td><td>2</td><td>3</td></tr><tr><td>B</td><td>0</td><td>0</td><td>2.5</td><td>2.5</td></tr><tr><td>C</td><td>0</td><td>2.5</td><td>0</td><td>4</td></tr><tr><td>D</td><td>2</td><td>4</td><td>0</td><td>0</td></tr></table>		P	Q	R	S	A	1	0	2	3	B	0	0	2.5	2.5	C	0	2.5	0	4	D	2	4	0	0	M1 A1ft	1.1b 1.1b																									
	P	Q	R	S																																																
A	1	0	2	3																																																
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D	2	4	0	0																																																
A – Q, B – P, C – R, D – S	A1ft	2.2a																																																		
	(5 marks)																																																			

M1: simplifying the initial matrix by reducing rows and then columns

A1: cao

M1: develop an improved solution – need to see one double covered +e; one uncovered –e; and one single covered unchanged. 3 lines to 4 lines needed

A1ft: allow follow through from one numerical slip only during row/column reduction

A1ft: dependent on all previous M marks and one A mark – to deduce the optimal allocation from the location of the zeros in the table

2.

Mark scheme (a)

Scheme	Marks	AO
Add an additional dummy column with equal values (e.g. 0) to create a square array	B1	3.5c
Input a suitable large number (e.g. 100 but > 44) in cells BQ and ES	B1	1.1b
	(2)	

Notes

B1: Explain the need to add a dummy column (to create a square array) – must mention that it is a column being added (but ignore any mention of the values being added to this column)

B1: Input a large value in cells BQ and ES (just saying ‘input a large value in the empty (oe) cells’ is sufficient – we do not need to know at this stage what is meant by ‘large’)

Mark scheme (b)

Scheme	Marks	AO																														
e.g. <table><tr><td></td><td>P</td><td>Q</td><td>R</td><td>S</td><td>X</td></tr><tr><td>A</td><td>38</td><td>39</td><td>37</td><td>37</td><td>0</td></tr><tr><td>B</td><td>39</td><td>100</td><td>39</td><td>40</td><td>0</td></tr><tr><td>C</td><td>41</td><td>44</td><td>40</td><td>42</td><td>0</td></tr><tr><td>D</td><td>40</td><td>41</td><td>39</td><td>38</td><td>0</td></tr></table>		P	Q	R	S	X	A	38	39	37	37	0	B	39	100	39	40	0	C	41	44	40	42	0	D	40	41	39	38	0	B1	1.1b
	P	Q	R	S	X																											
A	38	39	37	37	0																											
B	39	100	39	40	0																											
C	41	44	40	42	0																											
D	40	41	39	38	0																											

E	36	39	41	100	0
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Reducing (rows and) columns gives

	P	Q	R	S	X
A	2	0	0	0	0
B	3	61	2	3	0
C	5	5	3	5	0
D	4	2	2	1	0
E	0	0	4	63	0

M1

1.1b

Three lines required to cover the zeros hence solution is not optimal

	P	Q	R	S	X
A	2	0	0	0	1
B	2	60	1	2	0
C	4	4	2	4	0
D	3	1	1	0	0
E	0	0	4	63	1

M1

1.1b

Four lines required to cover the zeros hence solution is not optimal
e.g.

	P	Q	R	S	X
A	2	0	0	0	2
B	1	59	0	1	0
C	3	3	1	3	0
D	3	1	1	0	1
E	0	0	4	63	2

M1

1.1b

or

	P	Q	R	S	X
A	2	0	0	1	2
B	1	59	0	2	0
C	3	3	1	4	0
D	2	0	0	0	0
E	0	0	4	64	2

Five lines required to cover the zeros hence solution is optimal	B1	2.4
Allocation: A to Q, B to R, D to S, and E to P (C does no task)	A1	2.2a
	(6)	

Notes

B1: Mark awarded when both steps complete (addition of extra column and large values (> 44) in cells BQ and ES)

M1: Simplifying the initial matrix by reducing (rows and) columns – allow at most 2 slips

M1: Develop an improved solution – need to see one double covered +e; one uncovered –e; and one single covered unchanged. 3 lines needed to 4 lines needed

M1: Develop an improved solution – need to see one double covered +e; one uncovered –e; and one single covered unchanged. 4 lines needed to 5 lines needed (so getting to the optimal table)

B1: Dependent on two augmentations taking place (3 to 4 lines and then 4 to 5 lines). Either a correct statement(s) regarding the minimum number of lines to cover the zeros at each stage **or** a general statement that covers all augmentations.

In the first case, at each stage, they must state the number of lines (not just shown on the diagram), state whether it is optimal or not (so must use the word ‘optimal’) and mention ‘zeros’ at least once.

In the second case, they must state that until 5 lines cover the zeros then the solution is not optimal (or equivalent e.g. if there are 5 lines covering the zeros then the solution is optimal) – in this case they must show the lines.

Accept a hybrid of the two e.g. at each stage they could say whether it requires five lines or not but they would still have to mention ‘zeros’ at least once and make it clear at each augmentation whether it is optimal or not.

To award this mark we must see mention at least one mention of ‘zeros’ and the word ‘optimal’ being used.

A1: cso on final table + deduction of the correct allocation

Special case in (b) – if dashes are not replaced with larger values (e.g. values ≤ 44 or leaving dashes in place) then only the **M** marks are available in **(b)**

Mark scheme (c)

Scheme	Marks	AO
152 (mins)	B1	1.1b
	(1)	
	(9 marks)	

Notes

B1: cao

3.

Mark scheme (a)

Scheme	Marks	AO
(i) Subtracting each entry from a constant value e.g. ≥ 77 to convert from maximisation problem to minimisation	B1	3.5c
Add a sufficiently large number (> 12) to cells BQ and DR	B1	1.1b
(ii) e.g. $\begin{bmatrix} 12 & 5 & 8 & 2 \\ 6 & 100 & 9 & 12 \\ 7 & 8 & 4 & 0 \\ 4 & 7 & 100 & 6 \end{bmatrix}$	B1	1.1b
	(3)	

Notes

B1: correct reasoning for how to convert from maximisation to minimisation

B1: adding a large number (at least 13) to cells BQ and DR

B1: CAO

Mark scheme (b)

Scheme	Marks	AO
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Let x_{ij} be 0 or 1 $\begin{cases} 1 & \text{if worker } (i) \text{ does task } (j) \\ 0 & \text{otherwise} \end{cases}$	B1	3.3
where $i \in \{A, B, C, D\}$ and $j \in \{P, Q, R, S\}$	B1	2.5
minimise $12x_{AP} + 5x_{AQ} + 8x_{AR} + 2x_{AS} + 6x_{BP} + '100'x_{BQ} +$ $9x_{BR} + 12x_{BS}$ $+7x_{CP} + 8x_{CQ} + 4x_{CR} + 4x_{DP} + 7x_{DQ} + '100'x_{DR} +$ $6x_{DS}$	M1 A1	3.3 1.1b
Subject to $\sum x_{Aj} = 1, \sum x_{Bj} = 1, \sum x_{Cj} = 1, \sum x_{Dj} = 1$ $\sum x_{iP} = 1, \sum x_{iQ} = 1, \sum x_{iR} = 1, \sum x_{iS} = 1$	M1 A1	3.3 1.1b
	(6)	
	(9 marks)	

Notes

B1: defining x_{ij} correctly

B1: correct definition of the values that i and j can take

M1: Attempt at a 15 (or 16) term expression in terms of x (or their defined variable), coefficients 'correct', 2 'large' values included, condone 2 slips. If more than 2 errors in (a) ft exactly their table for this mark only.

A1: CAO including 'minimise' (or equivalent 14 term expression with 'maximise')

M1: At least four equations, each in three or four variables, unit coefficients, equal to 1 Do not accept inequalities

A1: CAO (all eight equations)

Maximise $65x_{AP} + 72x_{AQ} + 69x_{AR} + 75x_{AS}$
 $+71x_{BP} + 68x_{BR} + 65x_{BS}$
 $+70x_{CP} + 69x_{CQ} + 73x_{CR} + 77x_{CS}$
 $+73x_{DP} + 70x_{DQ} + 71x_{DS}$

Note: if maximising they must not include a large number for BQ and DR, so only 14 terms