

QQQ – Core Pure 2 – Chapter 8 – Modelling with Differential Equations (Version 2)

Total Marks: 24

(24 = Platinum, 22 = Gold, 19 = Silver, 16 = Bronze)

1. The movement of a particle is described by the simple harmonic equation

$$\ddot{x} = -25x$$

where x metres is the displacement of the particle at time t seconds, and $\ddot{x} \text{ m s}^{-2}$ is the acceleration of the particle.

The maximum displacement of the particle is 9 metres.

Find the maximum speed of the particle.

Circle your answer.

[1 mark]

- 15 m s^{-1}
- 45 m s^{-1}
- 75 m s^{-1}
- 135 m s^{-1}

(Total 1 marks)

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2. The solution of a second order differential equation is $f(t)$

The differential equation models heavy damping.

Which one of the statements below could be true?

Tick (✓) **one** box.

[1 mark]

- $f(t) = 2e^{-t} \cos(3t) + 5e^{-t} \sin(3t)$
- $f(t) = 3e^{-t} + 4te^{-t}$
- $f(t) = 7e^{-t} + 2e^{-2t}$
- $f(t) = 8e^{-t} \cos(3t - 0.1)$

(Total 1 marks)

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3. Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{dr}{dt} = 2 - \frac{r}{\alpha}$$

where α is a positive constant to be determined. **(2)**

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second. **(6)**

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer. **(1)**

(Total 9 marks)

4. A biologist is studying the effect of pesticides on crops. On a certain farm pesticide is regularly applied to a particular crop which grows in soil. Over time, pesticide is transferred between the crop and the soil at a rate which depends on the amount of pesticide in both the crop and the soil. The amount of pesticide in the crop after t days is x grams. The amount of pesticide in the soil after t days is y grams. Initially, when $t = 0$, there is no pesticide in either the crop or the soil.

At first it is assumed that no pesticide is lost from the system. The biologist further assumes that pesticide is added to the crop at a constant rate of k grams per day, where $k > 6$.

After collecting some initial data, the biologist suggests that for $t \geq 0$, this situation can be modelled by the following pair of first order linear differential equations.

$$\frac{dx}{dt} = -2x + 78y + k$$

$$\frac{dy}{dt} = 2x - 78y$$

(a) (i) Show that $\frac{d^2x}{dt^2} + 80\frac{dx}{dt} = 78k$. [2]

(ii) Determine the particular solution for x in terms of k and t . [7]

If more than 250 grams of pesticide is found in the crop, then it will fail food safety standards.

(iii) The crop is tested 50 days after the pesticide is first added to it.

Explain why, according to this model, the crop will fail food safety standards as a result of this test. [1]

Further data collection suggests that some pesticide decays in the soil and so is lost from the system. The model is refined in light of this data. The particular solution for x for this refined model is

$$x = k \left(20 - e^{-41t} \left(20 \cosh(\sqrt{1677} t) + \frac{819}{\sqrt{1677}} \sinh(\sqrt{1677} t) \right) \right).$$

(b) Given now that $k < 12$, determine whether the crop will fail food safety standards in the long run according to this refined model. [2]

In the refined model, it is still assumed that pesticide is added to the crop at a constant rate.

(c) Suggest a reason why it might be more realistic to model the addition of pesticide as not being at a constant rate. [1]

(Total 13 marks)

Solutions

1.

Scheme	Marks	AO
Circles 2nd answer	B1	2.2a
	(1 mark)	

Typical solution

$$45 \text{ m s}^{-1}$$

2.

Scheme	Marks	AO
Ticks correct answer	B1	1.2
	(1 mark)	

Typical solution

$$f(t) = 7e^{-t} + 2e^{-2t}$$

3.

Mark scheme (a)

Scheme	Marks	AO
Volume of paint = 30 litres therefore Rate of paint out = $3 \times \frac{r}{30}$ litres per second	M1	3.3
$\frac{dr}{dt} = 2 - \frac{r}{10}$	A1	1.1b
	(2)	

Notes

M1: Clearly identifies that Rate of paint out = $3 \times \frac{r}{\text{their volume}}$. It is a “show that” question so there must be clearly reasoning. Just answer with no reasoning scores M0.

A1: Puts all the components together to form the correct differential equation.

Mark scheme (b)

Scheme	Marks	AO
Rearranges $\frac{dr}{dt} + \frac{r}{10} = 2$ and attempts integrating factor IF = $e^{\int \frac{1}{10} dt} = \dots$	M1	3.1a
$re^{\frac{t}{10}} = \int 2e^{\frac{t}{10}} dt \Rightarrow re^{\frac{t}{10}} = \lambda e^{\frac{t}{10}} (+c)$	M1	1.1b
$re^{\frac{t}{10}} = 20e^{\frac{t}{10}} + c$	A1ft	1.1b
$t = 0, r = 10 \Rightarrow c = \dots$	M1	3.4
$r = \frac{20e^{\frac{t}{10}} - 10}{e^{\frac{t}{10}}} = 15$ rearranges to achieve $e^{\frac{t}{10}} = \alpha$ and solves to find a value for t or $r = 20 - 10e^{-\frac{t}{10}} = 15$ rearranges to achieve $e^{-\frac{t}{10}} = \beta$ and solves to find a value for t	M1	3.4
$t = \text{awrt } 7 \text{ seconds}$	A1	2.2b
	(6)	

Notes

(In these notes a stands for the constant α in the differential equation.)

M1: Identifies as a first order differential equation and finds the integrating factor or separates the variables and integrates. Allow if there are sign slips in rearranging (e.g. to $\frac{dr}{dt} - \frac{r}{10} = 2$) or in the integrating factor and allow with their value for a or with a as an unknown.

M1: Multiplies through by the IF and attempts to integrate or integrates to the form

$$\lambda \ln(2a - r) = \frac{1}{a}t + c \text{ oe}$$

A1ft: Correct integration, including constant of integration. Follow through on their value of a , but not sign slips from rearrangement. So allow for $re^{\frac{t}{a}} = 2ae^{\frac{t}{a}} + c$ or $-\ln(2a - r) = \frac{1}{a}t + c$ oe with a or their a .

M1: Uses the initial conditions to find the constant of integration. Must see substitution or can be implied by the correct value for their equation. Allow for finding in terms of a if separation of variables used.

M1: Sets $r = 15$, achieves $e^{\frac{t}{10}} = \alpha > 0$ or $e^{-\frac{t}{10}} = \beta > 0$ as appropriate and solves to find a value for t . Separates the variable method sets $r = 15$ and rearranges to find a value for t . **Note:** For this mark a

value of a is needed, but need not be the correct one.

Also: $t = \text{awrt } 7$ seconds from fully correct work.

Alternative: separating the variables

Scheme	Marks	AO
Separates the variables $\int \frac{1}{20-r} dr = \frac{1}{10} dt \Rightarrow \dots$	M1	3.1a
Integrates to the form $\lambda \ln(20-r) = \frac{1}{10}t (+c)$	M1	1.1b
$-\ln(20-r) = \frac{1}{10}t + c$	A1ft	1.1b
$t = 0, r = 10 \Rightarrow c = \dots$	M1	3.4
$-\ln(20-15) = \frac{1}{10}t - \ln 10$ Leading to a value for t	M1	3.4
$t = \text{awrt } 7$ seconds	A1	2.2b
	(6)	

Mark scheme (c)

Scheme	Marks	AO
The model predicts 7 seconds but it actually takes 9 seconds so (over) 2 seconds out (over 20%), therefore it is not a good model	B1ft	3.5a
	(1)	
	(9 marks)	

Notes

B1ft: See scheme, follow through on their answer to part (b). Accept any reasonable comparative comment but must have a reason, not just a statement of good or not good. So e.g. look for finding the difference between their answer and 9, or the percentage difference. If their answer is close to 9, then accept a conclusion of being a good model if a suitable reason is given. May substitute 9 into their equation and obtain a value to compare with 15 and make a similar conclusion.

4.

Mark scheme (a)

Scheme	Marks	AO
(i) $\frac{d^2x}{dt^2} = -2\frac{dx}{dt} + 78(2x - 78y)$ $= -2\frac{dx}{dt} + 156x - 78\left(\frac{dx}{dt} + 2x - k\right)$	M1	3.3
$\Rightarrow \frac{d^2x}{dt^2} = -80\frac{dx}{dt} + 78k$ $\Rightarrow \frac{d^2x}{dt^2} + 80\frac{dx}{dt} = 78k$	A1	1.1
	[2]	

Notes

M1: Differentiate $\frac{dx}{dt}$ and substitute $\frac{dy}{dt}$

A1: AG convincingly shown after second substitution.

Scheme	Marks	AO
(ii) $\lambda^2 + 80\lambda = 0$ $\Rightarrow \lambda = 0, -80$	M1	1.1
Complementary function is $x = A + Be^{-80t}$	A1	2.2a
Trial function is $x = at (+b)$ $\frac{dx}{dt} = a, \frac{d^2x}{dt^2} = 0$ $\Rightarrow 80a = 78k \Rightarrow a = \frac{39}{40}k$	M1	3.1a
$\Rightarrow$ GS is $x = A + Be^{-80t} + \frac{39}{40}kt$	A1	1.1
When $t = 0, x = 0 \Rightarrow A + B = 0$	M1	3.3
$\frac{dx}{dt} = -80Be^{-80t} + \frac{39}{40}k$ When $t = 0, x = 0, y = 0 \Rightarrow \frac{dx}{dt} = 0 + 0 + k$ $\Rightarrow k = -80B + \frac{39}{40}k$ $\Rightarrow B = -\frac{1}{3200}k, A = \frac{1}{3200}k$	M1	3.1a

Particular solution: $x = \frac{k}{3200} (1 - e^{-80t} + 3120t)$	A1	3.3
	[7]	

Notes

M1: Attempt to solve auxiliary equation for their DE

M1: Correct (or recovered) trial function for their CF.

A1: For GS

M1: Using $t = 0$ and $x = 0$ in *their* GS to find an equation in A and B .

M1: Differentiating and using $t = 0, x = 0, (y = 0), \dot{x} = k$ to find another equation in (A and) B .

Could also use $\dot{x} + \dot{y} = k$ and $x + y = kt$.

A1: N.B. $\frac{3120}{3200} = \frac{39}{40}$

Alternative method for first 4 marks

Scheme	Marks
$\dot{x} + 80x = 78kt + c$ $e^{\int 80 dt}(\dot{x} + 80x) = (78kt + c)e^{\int 80 dt}$ $\frac{d}{dt}(e^{80t}x) = (78kt + c)e^{80t}$	M1
$e^{80t}x = \int (78kt + c)e^{80t} dt + d$	M1
$e^{80t}x = \frac{(78kt + c)e^{80t}}{80} - \int \frac{78k}{80}e^{80t} dt + d$	M1
$e^{80t}x = \frac{(78kt + c)e^{80t}}{80} - \frac{39k}{3200}e^{80t} + d$ $x = \frac{39}{40}kt + \frac{c}{80} - \frac{39k}{3200} + de^{-80t}$ $= A + Be^{-80t} + \frac{39}{40}kt$	A1

M1: Integrating $\ddot{x} + 80\dot{x} = 78k$ wrt t and then using integrating factor

M1: Integrating wrt t

M1: Integrating by parts ($78kt + c$ may be separated)

A1: GS of form $x = A + Be^{-80t} + \frac{39}{40}kt$

Alternative method for this M mark

Scheme	Marks
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$78y = \dot{x} + 2x - k$ $= -78Be^{-80t} - \frac{1}{40}k + 2A + \frac{78}{40}kt$ When $t = 0, x = 0, y = 0$ $0 = 2A - 78B - \frac{1}{40}k$	M1
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M1: Finding GS for y and using $x = 0, y = 0$ to find another equation for A and B

Another alternative method for Q8(a)(ii) for first 4 marks

Scheme	Marks
D.E. is $\frac{dy}{dt} + 80y = 78k$ where $y = \frac{dx}{dt}$ I.F. $e^{80t} \Rightarrow e^{80t} \frac{dy}{dt} + 80e^{80t}y = 78ke^{80t}$ $\Rightarrow \frac{d}{dt}(e^{80t}y) = 78ke^{80t}$	M1
$\Rightarrow e^{80t}y = \frac{39k}{40}e^{80t} + A \Rightarrow y = \frac{39k}{40} + Ae^{-80t}$	A1
$\Rightarrow \frac{dx}{dt} = \frac{39k}{40} + Ae^{-80t} \Rightarrow x = \frac{39kt}{40} - \frac{A}{80}e^{-80t} + B$	M1 A1

Mark scheme for Q8(a)(ii) if candidate solves for y , rather than x

Scheme	Marks
$\ddot{y} = 2\dot{x} - 78\dot{y} = 2(-2x + 78y + k) - 78\dot{y}$ $\ddot{y} = -2(\dot{y} + 78y) + 156y + 2k - 78\dot{y}$ $\ddot{y} + 80\dot{y} = 2k$ $\lambda^2 + 80\lambda = 0$ $\lambda = 0, -80$	M1
Complementary function is $y = A + Be^{-80t}$	M1
Trial function is $y = at$ $\dot{y} = a, \ddot{y} = 0$ So $80a = 2k \Leftrightarrow a = \frac{1}{40}k$ General solution is $y = A + Be^{-80t} + \frac{1}{40}kt$	M1
$\dot{y} = -80Be^{-80t} + \frac{1}{40}k$ $x = \frac{1}{2}(\dot{y} + 78y)$ $= 39A - Be^{-80t} + \frac{39}{40}kt + \frac{1}{80}k$	A1

When $t = 0, x = 0, y = 0, \dot{y} = 0 + 0 = 0$ $0 = A + B$ $0 = 39A - B + \frac{1}{80}k$ $0 = -80B + \frac{1}{40}k$	M1 M1
$B = \frac{1}{3200}k, A = -\frac{1}{3200}k$ Particular solution for x is $x = -\frac{39}{3200}k - \frac{1}{3200}ke^{-80t} + \frac{39}{40}kt + \frac{1}{80}k$ $= \frac{1}{3200}k(1 - e^{-80t} + 3120t)$	A1
	[7]

M1: Auxiliary equation for their DE

M1: Correct trial function for their CF

Note the GS for x in the main mark scheme achieves **A1** but here **A1** is not awarded until GS for x found.

A1: GS of form $x = C + De^{-80t} + \frac{39}{40}kt$

M1 M1: Using two of $x = y = \dot{y} = 0$ to find simultaneous equations for A and B (**M1** for each equation).

Alternative method (solving for y)

Scheme	Marks
$\dot{y} + 80y = 2kt + c$ $e^{\int 80 dt}(\dot{y} + 80y) = (2kt + c)e^{\int 80 dt}$ $\frac{d}{dt}(e^{80t}y) = (2kt + c)e^{80t}$	M1
$e^{80t}y = \int (2kt + c)e^{80t} dt + d$	M1
$e^{80t}y = \frac{(2kt + c)e^{80t}}{80} - \int \frac{2}{80}ke^{80t} dt + d$ $e^{80t}y = \frac{(2kt + c)e^{80t}}{80} - \frac{1}{3200}ke^{80t} + d$ $y = \frac{1}{40}kt + \frac{1}{80}c - \frac{1}{3200}k + de^{-80t}$ $= A + Be^{-80t} + \frac{1}{40}kt$	M1

M1: Integrating $\ddot{y} + 80\dot{y} = 2k$ wrt t and then using integrating factor (corrected from the printed mark scheme, which reads $\ddot{y} + 78\dot{y} = 2k$)

M1: Integrating wrt t

M1: Integrating by parts

Scheme	Marks	AO
(iii) When $t = 50, x > \frac{156000k}{3200}$ (because $e^{-4000} < 1$ so $1 - e^{-4000} > 0$) i.e. $x > 48.75k > 292.5$ for $k > 6$ i.e. $x > 250$ so fails safety food standards	B1	3.4
	[1]	

Notes

B1: Use of $t = 50$ must be seen to give a term in k

Must reference $k > 6$.

Condone the idea of ignoring exponential term as negligible

Sight of $292.5 > 250$ earns the mark

Mark scheme (b)

Scheme	Marks	AO
$\sqrt{1677} = 40.95 \dots < 41$ $\Rightarrow e^{-41t} (a \cosh(\sqrt{1677}t) + b \sinh(\sqrt{1677}t))$ contains only negative exponentials when expanded	M1	3.4
so as $t \rightarrow \infty$, the exponential/hyperbolic parts $\rightarrow 0$ so $x \rightarrow 20k < 240 < 250$ since $k < 12$ so yes, food safety standards are met in the long run.	A1	3.2a
	[2]	

Notes

M1: Turning function into exponentials. Could see eg $e^{-0.0488t}$ and $e^{-81.95t}$. Condone error(s) in coefficients.

Allow argument such as e^{-41t} dominates

A1: Argument must be complete and correct but could be based on sufficiently large values of t rather than formal limiting process.

SC B1 argument that $20k = 240 < 250$ if 2nd term is assumed to tend to 0

Mark scheme (c)

Scheme	Marks	AO
Pesticide is likely to be added periodically, eg, during the day; or depending on the weather/time of year; or only when it's needed; or the amount of pesticide added changes as the amount of crop changes/grows or if it is subject to pest attack	B1	3.5a
	[1]	

Notes

B1: The idea that regularly is not the same as continuously (and may not even mean “at a constant average rate”).

Other sensible answers possible, but must be in context.