

QQQ – Core Pure 2 – Chapter 6 – Hyperbolic Functions (Version 2)

Total Marks: 25

(25 = Platinum, 23 = Gold, 20 = Silver, 17 = Bronze)

1. Prove the identity

$$\cosh^2 x - \sinh^2 x = 1$$

[2 marks]

(Total 2 marks)

AQA AS June 2021 Paper 1 Q6

2.

- (a) Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that, for $x \in \mathbb{R}$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

(2)

- (b) Hence, given that $-1 < \theta < 1$, prove that

$$\operatorname{artanh} \theta = \frac{1}{2} \ln \left(\frac{1 + \theta}{1 - \theta} \right)$$

(3)

(Total 5 marks)

Edexcel FP3 June 2018 Q1

3. Find the value of

$$\int_1^5 \frac{1}{\sqrt{x^2 + 6x + 5}} dx$$

Give your answer in the form

$$\ln(8 + a\sqrt{3} + b\sqrt{5} + c\sqrt{15})$$

where a , b and c are integers.

[5 marks]

(Total 5 marks)

AQA A2 June 2025 Paper 2 Q12

4. Solve the equation

$$2 \cosh^2 x - 3 \sinh x = 1$$

giving your answers in terms of natural logarithms.

(6)

(Total 6 marks)

Edexcel FP3 June 2015 Q1

5. The curve C has equation

$$y = 31 \sinh x - 2 \sinh 2x \quad x \in \mathbb{R}$$

Determine, in terms of natural logarithms, the exact x coordinates of the stationary points of C .

(7)

(Total 7 marks)

Edexcel A2 October 2020 Paper 2 Q1

Solutions

1.

Scheme	Marks	A0
Recalls exponential definitions of hyperbolic functions and substitutes $\frac{e^x + e^{-x}}{2}$ and $\frac{e^x - e^{-x}}{2}$ Condone $\cosh x$ replaced with $\frac{e^x - e^{-x}}{2}$ and $\sinh x$ replaced with $\frac{e^x + e^{-x}}{2}$	M1	1.2
Completes a fully correct proof to reach the required result.	R1	2.1
	(2 marks)	

Typical solution

$$\begin{aligned}
 \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\
 &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
 &= \frac{4}{4} \\
 &= 1
 \end{aligned}$$

2.

Mark scheme (a)

Scheme	Marks
$\left(\tanh x = \frac{\sinh x}{\cosh x} \right) = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} \text{ or } \frac{\frac{e^{2x} - 1}{2e^x}}{\frac{e^{2x} + 1}{2e^x}}$	M1
$= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}^*$	A1*
	(2)

Notes

M1: Substitutes the correct exponential forms. Note that the $\tanh x = \frac{\sinh x}{\cosh x}$ may be implied.

A1*: Correct proof with no errors or omissions or notational errors such as using sin for sinh

Note that the question says “starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials” so:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \text{ scores M1A1}$$

BUT

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \text{ scores MOA0 as } \sinh x \text{ and } \cosh x \text{ have not been defined}$$

Mark scheme (b)

Scheme	Marks
$y = \operatorname{artanh} \theta \Rightarrow \tanh y = \theta \Rightarrow \theta = \frac{e^{2y} - 1}{e^{2y} + 1}$ $\theta (e^{2y} + 1) = e^{2y} - 1 \Rightarrow e^{2y}(\theta - 1) = -1 - \theta \Rightarrow e^{2y} = \frac{1 + \theta}{1 - \theta}$ M1 for setting $\theta = \frac{e^{2y} - 1}{e^{2y} + 1}$ or any other variables for θ and y e.g. $y = \frac{e^{2x} - 1}{e^{2x} + 1}$ and uses correct processing (allow sign errors only) to make e^{2y} or e^y the subject	M1
$e^{2y} = \frac{1 + \theta}{1 - \theta} \Rightarrow 2y = \ln \left(\frac{1 + \theta}{1 - \theta} \right)$	dM1
$y = \frac{1}{2} \ln \left(\frac{1 + \theta}{1 - \theta} \right) \text{ * or } \frac{1}{2} \ln \frac{1 + \theta}{1 - \theta} \text{ *}$ Correct completion with no errors. Must be in terms of θ for this mark but allow “mixed” variables for the M’s. This mark should be withheld if there are any errors such as the appearance of a “tan” instead of “tanh” and/or missing variables. The proof does need to convey that $\operatorname{artanh} \theta = \frac{1}{2} \ln \left(\frac{1 + \theta}{1 - \theta} \right)$ So if y has been defined as $\operatorname{artanh} \theta$ and the proof ends $y = \frac{1}{2} \ln \left(\frac{1 + \theta}{1 - \theta} \right)$, this is acceptable. So must be in terms of θ for the A mark but allow other variables to be used for the M’s. Allow $\operatorname{arctanh}$, artanh , $\tanh^{-1}$ etc. for the inverse	A1*
	(3)
	(5 marks)

Notes

dM1: Removes e correctly by taking ln's. **Dependent on the first method mark.**

(b) Alternative

Scheme	Marks
$\operatorname{artanh} \theta = \frac{1}{2} \ln \left(\frac{1+\theta}{1-\theta} \right) \Rightarrow \theta = \tanh \left(\frac{1}{2} \ln \left(\frac{1+\theta}{1-\theta} \right) \right)$	
$\theta = \frac{e^{\ln \left(\frac{1+\theta}{1-\theta} \right)} - 1}{e^{\ln \left(\frac{1+\theta}{1-\theta} \right)} + 1}$	M1
$= \frac{\frac{1+\theta}{1-\theta} - 1}{\frac{1+\theta}{1-\theta} + 1} = \frac{1+\theta - 1 + \theta}{1+\theta + 1 - \theta} = \theta$	dM1
$\Rightarrow \operatorname{artanh} \theta = \frac{1}{2} \ln \left(\frac{1+\theta}{1-\theta} \right) *$ Allow $\frac{1}{2} \ln \frac{1+\theta}{1-\theta} *$	A1*
	(3)

M1: Uses part (a) to express θ in terms of e

dM1: Removes e's and ln's correctly. **Dependent on the first method mark.**

A1*: Obtains $\theta = \theta$ with no errors and makes a conclusion. Must be in terms of θ for this mark but allow a different variable for the M's. This mark should be withheld if there are any errors such as the appearance of a "tan" instead of "tanh" and/or missing variables.

Attempts that assume $\operatorname{artanh} \theta = \frac{\operatorname{arsinh} \theta}{\operatorname{arcosh} \theta}$ score no marks in (b)

3.

Scheme	Marks	AO
Completes the square to obtain $(x+p)^2 - q^2$	M1	3.1a
Integrates to obtain $\cosh^{-1} \left(\frac{x+p}{q} \right)$ or $\ln \left(x+p + \sqrt{(x+p)^2 - q^2} \right)$	A1F	1.1b
Substitutes the limits and deduces $\ln(4 + \sqrt{15}) - \ln(2 + \sqrt{3})$ or $\ln(8 + \sqrt{60}) - \ln(4 + \sqrt{12})$	A1	2.2a

Uses laws of logs to simplify result of their integration.	M1	1.1a
Completes a reasoned argument to obtain $\ln(8 - 4\sqrt{3} - 3\sqrt{5} + 2\sqrt{15})$	R1	2.1
	(5 marks)	

Typical solution

$$\begin{aligned}
 \int_1^5 \frac{1}{\sqrt{x^2 + 6x + 5}} dx &= \int_1^5 \frac{1}{\sqrt{(x+3)^2 - 2^2}} dx \\
 &= \left[\cosh^{-1} \left(\frac{x+3}{2} \right) \right]_1^5 \\
 &= \cosh^{-1} 4 - \cosh^{-1} 2 \\
 &= \ln(4 + \sqrt{15}) - \ln(2 + \sqrt{3}) \\
 &= \ln \left(\frac{4 + \sqrt{15}}{2 + \sqrt{3}} \right) \\
 &= \ln \left(\left(\frac{4 + \sqrt{15}}{2 + \sqrt{3}} \right) \left(\frac{2 - \sqrt{3}}{2 - \sqrt{3}} \right) \right) \\
 &= \ln \left(\frac{8 - 4\sqrt{3} + 2\sqrt{15} - \sqrt{45}}{4 - 3} \right) \\
 &= \ln(8 - 4\sqrt{3} - 3\sqrt{5} + 2\sqrt{15})
 \end{aligned}$$

4.

Scheme	Marks
$2(1 + \sinh^2 x) - 3 \sinh x = 1$	M1
$2 \sinh^2 x - 3 \sinh x + 1 = 0$	A1
$(2 \sinh x - 1)(\sinh x - 1) = 0$	M1
$\sinh x$ or $\frac{e^x - e^{-x}}{2} = \frac{1}{2}$ or 1	A1
$x = \ln \frac{1}{2} (1 + \sqrt{5}), \ln (1 + \sqrt{2})$	A1, A1 M1A1 on ePEN
	(6)
	(6 marks)

Notes

M1: Attempt to use $\cosh^2 x = 1 + \sinh^2 x$

A1: Correct 3 term quadratic. The “= 0” may be implied by their attempt to solve.

M1: Attempts to solve their 3TQ = 0 leading to $\sinh x = \dots$ (= 0 may be implied)

A1: Both values correct

A1: $x = \ln \frac{1}{2} (1 + \sqrt{5})$ or $\ln (1 + \sqrt{2})$ oe

A1: $x = \ln \frac{1}{2} (1 + \sqrt{5})$ and $\ln (1 + \sqrt{2})$ oe and **no other values**

Allow equivalent answers e.g. $\ln \left(\frac{1}{2} + \sqrt{\frac{5}{4}} \right)$, $\ln \left(\frac{1}{2} + \sqrt{1 + \frac{1}{4}} \right)$ and allow awrt 3SF accuracy e.g.

$\ln 1.62$, $\ln 2.41$

Alternative

Scheme	Marks
$2 \left(\frac{e^x + e^{-x}}{2} \right)^2 - 3 \left(\frac{e^x - e^{-x}}{2} \right) = 1$	M1
$e^{4x} - 3e^{3x} + 3e^x + 1 = 0$	A1
$(e^{2x} - e^x - 1)(e^{2x} - 2e^x - 1) = 0 \Rightarrow e^x = \dots$	M1
$e^x = \frac{1 + \sqrt{5}}{2}, \frac{2 + \sqrt{8}}{2}$	A1
$x = \ln \frac{1}{2} (1 + \sqrt{5}), \ln (1 + \sqrt{2})$	A1, A1 M1A1 on ePEN

M1: Substitutes **correct** definitions for $\sinh x$ and $\cosh x$ in terms of exponentials

A1: Correct quartic in e^x

M1: Solves their quartic as far as $e^x = \dots$ For the **correct quartic** there must be a recognisable attempt to solve e.g. the product of two 3TQ's in e^x or if answers only are given, they must be correct (1.62, 2.41, and possibly (-0.618, -0.414)). For an **incorrect quartic** there must be a recognisable attempt to solve a quartic with at least 4 terms.

A1: Correct values for e^x . Allow $e^x = \frac{1 \pm \sqrt{5}}{2}, \frac{2 \pm \sqrt{8}}{2}$ **but no incorrect values**. Allow awrt 1.62, 2.41

A1: $x = \ln \frac{1}{2} (1 + \sqrt{5})$ or $\ln (1 + \sqrt{2})$ oe

A1: $x = \ln \frac{1}{2} (1 + \sqrt{5})$ and $\ln (1 + \sqrt{2})$ oe and **no other values**. allow awrt 3SF accuracy e.g.

$\ln 1.62$, $\ln 2.41$

5.

Scheme	Marks	AO
$\frac{dy}{dx} = 31 \cosh x - 4 \cosh 2x$	B1	1.1b
$\frac{dy}{dx} = 31 \cosh x - 4 (2 \cosh^2 x - 1)$	M1	3.1a
$8 \cosh^2 x - 31 \cosh x - 4 = 0$	A1	1.1b
$(8 \cosh x + 1)(\cosh x - 4) = 0 \Rightarrow \cosh = \dots$	M1	1.1b
$\cosh x = 4, \left(-\frac{1}{8}\right)$	A1	1.1b
$\cosh x = \alpha \Rightarrow x = \ln(\alpha + \sqrt{\alpha^2 - 1})$ or $\ln(\alpha - \sqrt{\alpha^2 - 1})$ or $-\ln(\alpha + \sqrt{\alpha^2 - 1})$ or $\ln(\alpha - \sqrt{\alpha^2 - 1})$ or $\frac{e^x + e^{-x}}{2} = 4 \Rightarrow e^{2x} - 8e^x + 1 = 0 \Rightarrow e^x = \dots \Rightarrow x = \ln(\dots)$	M1	1.2
$\pm \ln(4 + \sqrt{15})$ or $\ln(4 \pm \sqrt{15})$	A1	2.2a
	(7)	

Alternative

Scheme	Marks	AO
$\frac{dy}{dx} = 31 \cosh x - 4 \cosh 2x$ or $31 \left(\frac{e^x + e^{-x}}{2}\right) - 4 \left(\frac{e^{2x} + e^{-2x}}{2}\right)$	B1	1.1b
Using $\cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$ and $\sinh x = \left(\frac{e^x - e^{-x}}{2}\right)$ as required $\Rightarrow 31 \left(\frac{e^x + e^{-x}}{2}\right) - 4 \left(\frac{e^{2x} + e^{-2x}}{2}\right) = 0$ leading to $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ o.e.	M1 A1	3.1a 1.1b
Solves $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ $\Rightarrow e^x = \dots$	M1	1.1b

$e^x = 4 \pm \sqrt{15}$ or awrt 7.87, 0.13	A1	1.1b
$x = \ln(b)$ where b is a real exact value	M1	1.2
$\ln(4 \pm \sqrt{15})$	A1	2.2a
	(7)	
	(7 marks)	

Notes

B1: Correct differentiation

M1: Identifies a correct approach by using a correct identity to make progress to obtain a quadratic in $\cosh x$

A1: Correct 3 term quadratic obtained

M1: Solves their 3TQ

A1: Correct values (may only see 4 here)

M1: Correct process to reach at least one value for x from their $\cosh x$

A1: Deduces the correct 2 values with no incorrect values or work involving $\cosh x = -\frac{1}{8}$

Alternative

B1: Correct differentiation

M1: Using the exponential form for $\cosh x$, and $\sinh x$ if required, and forms a quartic equation for e^x with all terms simplified and all on one side

A1: Correct quartic equation for e^x

M1: Solves their quartic equation in e^x

A1: Correct values to two decimal places or exact values

M1: $x = \ln(b)$ where b is a real exact value

A1: Deduces the correct 2 values only

(corrected from the printed mark scheme: $e^{2x} - 8e^x + 1 = 0$ is printed as $e^{2x} - 8e^x + 7 = 0$)