

QQQ – Core Pure 2 – Chapter 6 – Hyperbolic Functions (Version 1)

Total Marks: 25

(25 = Platinum, 23 = Gold, 20 = Silver, 17 = Bronze)

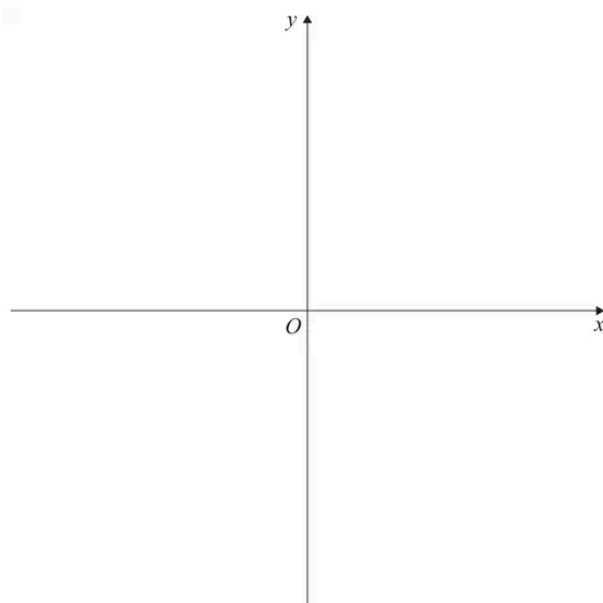
1.

(a) On the axes below, sketch the graph of

$$y = \cosh x$$

Indicate the value of any intercept of the curve with the axes.

[2 marks]



(b) Solve the equation

$$\cosh x = 2$$

Give your answers to three significant figures.

[2 marks]

(Total 4 marks)

2.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Determine the exact values of x for which

$$\sinh 2x = 3 \sinh x$$

(5)

(Total 5 marks)

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3.

(a) Given that $y = \cosh^{-1} \left(\frac{1}{3}x \right)$, find $\frac{dy}{dx}$ in terms of x . **[1]**

(b) Determine an equation of the normal to the curve $y = \cosh^{-1} \left(\frac{1}{3}x \right)$ at the point where $x = 5$.

Give your answer in the form $ax + by = c + \ln d$, where a, b, c and d are integers. **[4]**

(Total 5 marks)

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4.

(a) Find

$$\int \frac{1}{\sqrt{(4x^2 + 9)}} dx$$

(2)

(b) Use your answer to part (a) to find the exact value of

$$\int_{-3}^3 \frac{1}{\sqrt{(4x^2 + 9)}} dx$$

giving your answer in the form $k \ln(a + b\sqrt{5})$, where a and b are integers and k is a constant. **(3)**

(Total 5 marks)

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5.

- (a)** Using the exponential definitions of $\sinh x$ and $\cosh x$, prove that

$$\coth^{-1}(x) = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

[3 marks]

- (b)** Hence, solve the equation

$$\coth^{-1}(x) = -\ln 5$$

Give your answer in an exact form.

[3 marks]

(Total 6 marks)

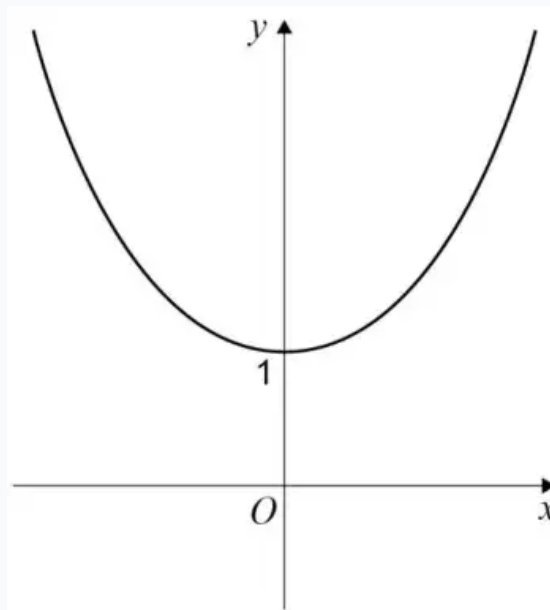
Solutions

1.

Mark scheme (a)

Scheme	Marks	AO
Draws the correct shape, approximately symmetrical about the y -axis and above the x -axis	B1	1.1b
Indicates one y -axis intercept at $(0, 1)$ Accept 1 clearly labelled on the y -axis.	B1	1.1b
	(2)	

Typical solution



Mark scheme (b)

Scheme	Marks	AO
Obtains at least one correct solution. Accept $\ln(2 + \sqrt{3})$ or $\ln(2 - \sqrt{3})$ or equivalent $\ln$ expression. Accept AWRT 1.32 or -1.32 Do not accept $\cosh^{-1} 2$	M1	1.1a
Obtains AWRT ± 1.32 with no other solutions given.	A1	1.1b

	(2)	
	(4 marks)	

Typical solution

$$x = \pm \cosh^{-1} 2$$

$$x = \pm 1.32 \text{ (3sf)}$$

2.

Scheme	Marks	AO
Uses the identity $\sinh 2x = 2 \sinh x \cosh x$ to find a value for $\sinh x$ or $\cosh x$ $2 \sinh x \cosh x = 3 \sinh x \Rightarrow 2 \sinh x \cosh x - 3 \sinh x = 0$ $\sinh x(2 \cosh x - 3) = 0 \Rightarrow \sinh x = \dots$ or $\cosh x = \dots$	M1	3.1a
$\sinh x = 0 \Rightarrow x = 0$	B1	1.1b
$\cosh x = \frac{3}{2} \Rightarrow x = \ln \left[\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1} \right]$ Alternatively $\cosh x = \frac{3}{2} \Rightarrow \frac{1}{2}(e^x + e^{-x}) \Rightarrow e^{2x} - 3e^x + 1 = 0$ $\Rightarrow e^x = \frac{3 + \sqrt{5}}{2}$ or $\frac{3 - \sqrt{5}}{2} \Rightarrow x = \ln \dots$	dM1	1.1b
$x = \ln \left[\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1} \right]$	A1	1.1b
$x = \pm \ln \left(\frac{3 + \sqrt{5}}{2} \right)$ or $x = \ln \left(\frac{3 \pm \sqrt{5}}{2} \right)$	A1	2.2a
	(5)	
	(5 marks)	

Notes

M1: Uses the identity $\sinh 2x = 2 \sinh x \cosh x$ and proceeds to find a value for $\sinh x$ or $\cosh x$

B1: $x = 0$

dM1: Uses the correct formula for $\operatorname{arcosh} x$ with their value of $\cosh x$ to find a value for x as a natural logarithm or alternatively gives awrt to 3sf. e.g. 0.962, -0.962 . You may need to check their workings.

Alternatively uses the exponential definition for $\cosh x$, forms and solves a quadratic for e^x leading to find a value for x as a natural logarithm. Usual rules apply for solving a quadratic by any means including using a calculator. You may need to check their workings. If correct answers are given without workings, you may award this mark.

A1: Deduces one correct exact value for x . Accept exact unsimplified equivalents.

A1: Deduces both correct exact values for x . Accept exact simplified equivalents. isw

Accept, for example $x = \pm \ln \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right)$ and $x = \pm \ln \left(\frac{3}{2} + \sqrt{\frac{5}{4}} \right)$

Also accept equivalent answers which you may need to check; their may be several alternatives.

e.g. Accept $x = \ln \left(\frac{2}{3 + \sqrt{5}} \right)$ for $x = \ln \left(\frac{3 - \sqrt{5}}{2} \right)$

Alternative method use of exponentials

Scheme	Marks	AO
Uses the identities $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$ $\frac{e^{2x} - e^{-2x}}{2} = 3 \left(\frac{e^x - e^{-x}}{2} \right)$ leading to a quartic equation for e^x (Note correct equation is $e^{4x} - 3e^{3x} + 3e^x - 1 = 0$)	M1	3.1a
$e^x = 1 \Rightarrow x = 0$	B1	1.1b
Solves quartic to find a value for e^x leading to $x = \ln \dots$	dM1	1.1b
$x = \ln \left[\frac{3}{2} + \sqrt{\left(\frac{3}{2} \right)^2 - 1} \right]$	A1	1.1b
$x = \pm \ln \left(\frac{3 + \sqrt{5}}{2} \right)$ or $x = \ln \left(\frac{3 \pm \sqrt{5}}{2} \right)$	A1	2.2a
	(5)	

M1: Uses correct identities such as $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$ to form a quartic equation for e^x

B1: $x = 0$

dM1: Solve their quartic equation to find a value for x (other than $x = 0$) via a correct method. If correct answers are given from their quartic equation without workings, you may award this mark. You may need to check their workings.

A1: Deduces one correct exact value for x , which may be unsimplified.

A1: Deduces both correct exact values for x isw

Note: Inexact answers for the quartic equation $e^x = 2.618$, 0.3819 leading to awrt 0.962, -0.962 scores maximum M1 B1 dM1 A0 A0

3.

Mark scheme (a)

Scheme	Marks	AO
$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 9}}$	B1	1.1
	[1]	

Notes

B1: Accept any equivalent (un-simplified) forms e.g. $\frac{1}{3} \times \frac{1}{\sqrt{\left(\frac{x}{3}\right)^2 - 1}}$ or $\frac{1}{\sqrt{x^2 - 3^2}}$. Invisible brackets

e.g. $\frac{1}{3} \times \frac{1}{\sqrt{\frac{x^2}{3} - 1}}$ is **B0** unless correctly recovered. ISW if correct un-simplified form is simplified incorrectly

Mark scheme (b)

Scheme	Marks	AO
$m_N = -4$	B1	1.1
$y = \ln \left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1} \right) \quad (= \ln 3)$	B1	1.1
$y - \ln 3 = -4(x - 5)$	M1	1.1
$4x + y = 20 + \ln 3$	A1	1.1
	[4]	

Notes

B1: Correct normal gradient (soi) – possibly BC

B1: For correct un-simplified (or simplified) y -coordinate in logarithmic form when $x = 5$ – ISW if simplified incorrectly

M1: Any complete correct method for the equation of a straight line with $x = 5$, their y coordinate (which must be of the form $\ln(a)$ where $a > 0$) and any non-zero gradient. If using $y = mx + c$ then must explicitly find c using $x = 5$, any non-zero m and their y which must be of the form as stated above (but allow errors in the evaluation of c)

A1: oe but must be integer values e.g. $8x + 2y = 40 + \ln 9$ and must be of the form $ax + by = c + \ln d$ e.g. allow $-\ln 3 - 20 = -y - 4x$

4.

Mark scheme (a)

Scheme	Marks
$k \operatorname{arsinh} \left(\frac{2x}{3} \right) (+c)$ or $k \ln \left[px + \sqrt{\left(p^2 x^2 + \frac{9}{4} p^2 \right)} \right] (+c)$	M1
$\frac{1}{2} \operatorname{arsinh} \left(\frac{2x}{3} \right) (+c)$ or $\frac{1}{2} \ln \left[px + \sqrt{\left(p^2 x^2 + \frac{9}{4} p^2 \right)} \right] (+c)$	A1
	(2)

Notes

Alternative for (a)

Scheme	Marks
$x = \frac{3}{2} \sinh u \Rightarrow \int \frac{1}{\sqrt{9 \sinh^2 u + 9}} \cdot \frac{3}{2} \cosh u \, du = k \operatorname{arsinh} \left(\frac{2x}{3} \right) (+c)$	M1
$\frac{1}{2} \operatorname{arsinh} \left(\frac{2x}{3} \right) (+c)$	A1

Mark scheme (b)

Scheme	Marks
So: $\frac{1}{2} \ln [6 + \sqrt{45}] - \frac{1}{2} \ln [-6 + \sqrt{45}] = \frac{1}{2} \ln \left[\frac{6 + \sqrt{45}}{-6 + \sqrt{45}} \right]$ Uses correct limits <u>and</u> combines logs	M1
$= \frac{1}{2} \ln \left[\frac{6 + \sqrt{45}}{-6 + \sqrt{45}} \right] \left[\frac{6 + \sqrt{45}}{6 + \sqrt{45}} \right] = \frac{1}{2} \ln \left[\frac{(6 + \sqrt{45})^2}{9} \right]$ Correct method to rationalise denominator (may be implied)	M1

Method must be clear if answer does not follow their fraction	
$= \ln[2 + \sqrt{5}] \quad (\text{or } \frac{1}{2} \ln[9 + 4\sqrt{5}])$	Alcso
	(3)
	(5 marks)

Notes

Note that the last 3 marks can be scored without the need to rationalise e.g.

$$2 \times \frac{1}{2} \left[\ln[2x + \sqrt{(4x^2 + 9)}] \right]_0^3 = \ln(6 + \sqrt{45}) - \ln 3 = \ln \left(\frac{6 + \sqrt{45}}{3} \right)$$

M1: Uses the limits 0 and 3 and doubles

M1: Combines Logs

A1: $\ln[2 + \sqrt{5}]$ oe

Alternative for (b)

Scheme	Marks
$\left[\frac{1}{2} \operatorname{arsinh} \left(\frac{2x}{3} \right) \right]_{-3}^3 = \frac{1}{2} \operatorname{arsinh} 2 - \frac{1}{2} \operatorname{arsinh} -2$	
$\frac{1}{2} \ln(2 + \sqrt{5}) - \frac{1}{2} \ln(\sqrt{5} - 2) = \frac{1}{2} \ln \left(\frac{2 + \sqrt{5}}{\sqrt{5} - 2} \right)$ Uses correct limits <u>and</u> combines logs	M1
$= \frac{1}{2} \ln \left(\frac{2 + \sqrt{5}}{\sqrt{5} - 2} \cdot \frac{\sqrt{5} + 2}{\sqrt{5} + 2} \right) = \frac{1}{2} \ln \left(\frac{2\sqrt{5} + 4 + 5 + 2\sqrt{5}}{5 - 4} \right)$ Correct method to rationalise denominator (may be implied) Method must be clear if answer does not follow their fraction	M1
$= \frac{1}{2} \ln[9 + 4\sqrt{5}]$	Alcso

5.

Mark scheme (a)

Scheme	Marks	AO
Uses definitions of sinh and cosh. PI $x = \frac{e^y + e^{-y}}{e^y - e^{-y}}$ or $x = \frac{e^{2y} + 1}{e^{2y} - 1}$	M1	1.1a

Rearranges to make e^{2y} or e^{2x} the subject. or Uses the quadratic equation formula	M1	3.1a
Completes a reasoned argument, starting from the full definitions of sinh and cosh, to obtain $\coth^{-1}(x) = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$ AG	R1	2.1
	(3)	

Typical solution

Let $y = \coth^{-1}(x)$

Then $x = \coth y$

$$x = \frac{\frac{1}{2}(e^y + e^{-y})}{\frac{1}{2}(e^y - e^{-y})} = \frac{e^{2y} + 1}{e^{2y} - 1}$$

$$x(e^{2y} - 1) = e^{2y} + 1$$

$$xe^{2y} - x = e^{2y} + 1$$

$$e^{2y}(x - 1) = x + 1$$

$$e^{2y} = \frac{x+1}{x-1}$$

$$2y = \ln \left(\frac{x+1}{x-1} \right)$$

$$y = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

$$\coth^{-1}(x) = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

Mark scheme (b)

Scheme	Marks	AO
Uses the result from part (a) to form an equation and applies at least one law of logs correctly	M1	3.1a
Obtains $\frac{x+1}{x-1} = \frac{1}{25}$ OE	A1	1.1b

Obtains $-\frac{13}{12}$ OE	A1	1.1b
	(3)	
	(6 marks)	

Typical solution

$$\frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) = -\ln 5 = \ln \frac{1}{5}$$

$$\ln \left(\frac{x+1}{x-1} \right) = 2 \ln \frac{1}{5} = \ln \left(\frac{1}{5} \right)^2$$

$$\frac{x+1}{x-1} = \frac{1}{25}$$

$$25(x+1) = x-1$$

$$25x + 25 = x - 1$$

$$x = -\frac{13}{12}$$