

QQQ – Core Pure 2 – Chapter 4 – Volumes of Revolution (Version 2)

Total Marks: 23

(23 = Platinum, 21 = Gold, 18 = Silver, 16 = Bronze)

1.

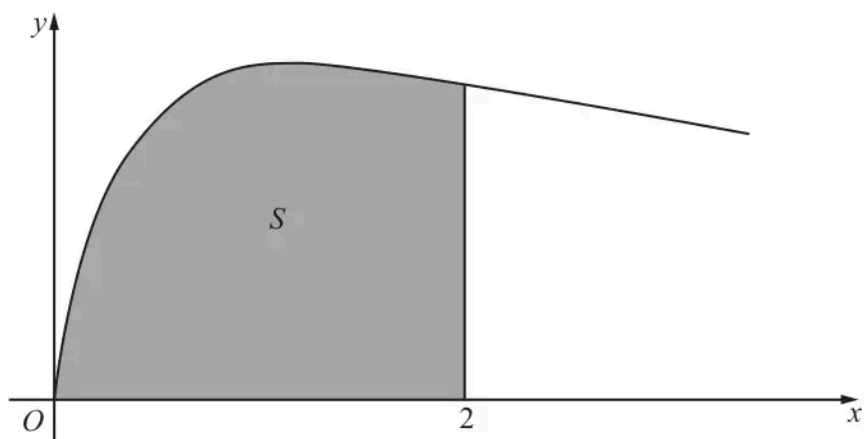


Figure 1

Figure 1 shows the curve with equation

$$y = \sqrt{\left(\frac{2x}{3x^2 + 4}\right)}, \quad x \geq 0$$

The finite region S , shown shaded in Figure 1, is bounded by the curve, the x -axis and the line $x = 2$

The region S is rotated 360° about the x -axis.

Use integration to find the exact value of the volume of the solid generated, giving your answer in the form $k \ln a$, where k and a are constants.

(5)

(Total 5 marks)

2.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

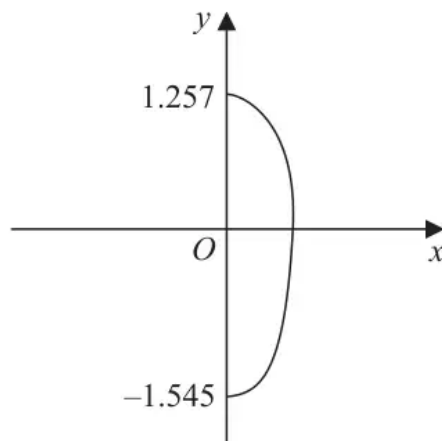


Figure 2

John picked 100 berries from a plant.

The largest berry picked was approximately 2.8 cm long.

The shape of this berry is modelled by rotating the curve with equation

$$16x^2 + 3y^2 - y \cos\left(\frac{5}{2}y\right) = 6 \quad x \geq 0$$

shown in Figure 2, about the y -axis through 2π radians, where the units are cm.

Given that the y intercepts of the curve are -1.545 and 1.257 to four significant figures,

(a) use algebraic integration to determine, according to the model, the volume of this berry. (6)

Given that the 100 berries John picked were then squeezed for juice,

(b) use your answer to part (a) to decide whether, in reality, there is likely to be enough juice to fill a 200 cm^3 cup, giving a reason for your answer. (2)

(Total 8 marks)

3.

- (a) Using the identity $\cos 2\theta = 1 - 2\sin^2 \theta$, find $\int \sin^2 \theta \, d\theta$. (2)

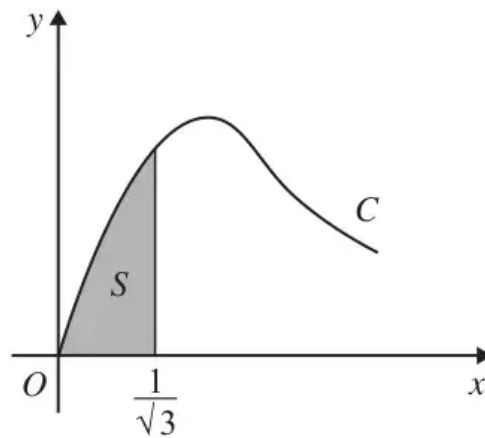


Figure 4

Figure 4 shows part of the curve C with parametric equations

$$x = \tan \theta, \quad y = 2 \sin 2\theta, \quad 0 \leq \theta < \frac{\pi}{2}$$

The finite shaded region S shown in Figure 4 is bounded by C , the line $x = \frac{1}{\sqrt{3}}$ and the x -axis. This shaded region is rotated through 2π radians about the x -axis to form a solid of revolution.

- (b) Show that the volume of the solid of revolution formed is given by the integral

$$k \int_0^{\frac{\pi}{6}} \sin^2 \theta \, d\theta$$

where k is a constant. (5)

- (c) Hence find the exact value for this volume, giving your answer in the form $p\pi^2 + q\pi\sqrt{3}$, where p and q are constants. (3)

(Total 10 marks)

Solutions

1.

Scheme	Marks
Volume = $\pi \int_0^2 \left(\sqrt{\left(\frac{2x}{3x^2 + 4} \right)} \right)^2 dx$ Use of $V = \pi \int y^2 dx$.	B1
$= (\pi) \left[\frac{1}{3} \ln(3x^2 + 4) \right]_0^2 \pm k \ln(3x^2 + 4)$	M1
$\frac{1}{3} \ln(3x^2 + 4)$	A1
$= (\pi) \left[\left(\frac{1}{3} \ln 16 \right) - \left(\frac{1}{3} \ln 4 \right) \right]$ Substitutes limits of 2 and 0 and subtracts the correct way round.	dM1
So Volume = $\frac{1}{3} \pi \ln 4$ $\frac{1}{3} \pi \ln 4$ or $\frac{2}{3} \pi \ln 2$	A1 oe isw
	(5)
	(5 marks)

Notes

NOTE: π is required for the B1 mark and the final A1 mark. It is not required for the 3 intermediate marks.

B1: For applying $\pi \int y^2$. Ignore limits and dx . This can be implied by later working, but the pi and $\int \frac{2x}{3x^2 + 4}$ must appear on one line somewhere in the candidate's working.

B1 can also be implied by a correct final answer. **Note:** $\pi \left(\int y \right)^2$ would be B0.

Working in x

M1: For $\pm k \ln(3x^2 + 4)$ or $\pm k \ln \left(x^2 + \frac{4}{3} \right)$ where k is a constant and k can be 1.

Note: M0 for $\pm kx \ln(3x^2 + 4)$.

Note: M1 can also be given for $\pm k \ln(p(3x^2 + 4))$, where k and p are constants and k can be 1.

A1: For $\frac{1}{3} \ln(3x^2 + 4)$ or $\frac{1}{3} \ln \left(\frac{1}{3} (3x^2 + 4) \right)$ or $\frac{1}{3} \ln \left(x^2 + \frac{4}{3} \right)$ or $\frac{1}{3} \ln(p(3x^2 + 4))$.

You may allow M1 A1 for $\frac{1}{3} \left(\frac{x}{x} \right) \ln(3x^2 + 4)$ or $\frac{1}{3} \left(\frac{2x}{6x} \right) \ln(3x^2 + 4)$

dM1: Substitutes limits of 2 and 0 and subtracts the correct way round. Working in decimals is fine for dM1.

A1: For either $\frac{1}{3} \pi \ln 4$, $\frac{1}{3} \pi \ln 4^\pi$, $\frac{2}{3} \pi \ln 2$, $\pi \ln 4^{\frac{1}{3}}$, $\pi \ln 2^{\frac{2}{3}}$, $\frac{1}{3} \pi \ln \left(\frac{16}{4} \right)$, $2\pi \ln \left(\frac{16^{\frac{1}{3}}}{4^{\frac{1}{6}}} \right)$, etc.

Note: $\frac{1}{3} \pi (\ln 16 - \ln 4)$ would be A0.

Working in u where $u = 3x^2 + 4$,

M1: For $\pm k \ln u$ where k is a constant and k can be 1.

Note: M1 can also be given for $\pm k \ln(pu)$, where k and p are constants and k can be 1.

A1: For $\frac{1}{3} \ln u$ or $\frac{1}{3} \ln 3u$ or $\frac{1}{3} \ln pu$.

dM1: Substitutes limits of 16 and 4 in u or limits of 2 and 0 in x and subtracts the correct way round.

A1: As above!

2.

Mark scheme (a)

Scheme	Marks	AO
$\frac{\pi}{16} \int_{-1.545}^{1.257} \left(6 - 3y^2 + y \cos \left(\frac{5}{2}y \right) \right) \{dy\}$ $\pi \int_{-1.545}^{1.257} \left(\frac{3}{8} - \frac{3}{16}y^2 + \frac{1}{16}y \cos \left(\frac{5}{2}y \right) \right) \{dy\}$	B1	1.1a
$\int x^2 dy = \frac{1}{16} \int 6 - 3y^2 + y \cos \left(\frac{5}{2}y \right) dy \rightarrow Ky - Ly^3 + \dots$	M1	1.1b
$\int y \cos \left(\frac{5}{2}y \right) dy = Ay \sin \left(\frac{5}{2}y \right) + B \cos \left(\frac{5}{2}y \right) (+c)$ $\left\{ \int y \cos \left(\frac{5}{2}y \right) dy = y \cdot \frac{2}{5} \sin \left(\frac{5}{2}y \right) - \int 1 \cdot \frac{2}{5} \sin \left(\frac{5}{2}y \right) dy = \frac{2}{5}y \sin \left(\frac{5}{2}y \right) + \frac{4}{25} \cos \left(\frac{5}{2}y \right) (+c) \right\}$	M1	3.1a
$\int x^2 dy = \frac{1}{16} \left(6y - y^3 + \frac{2}{5}y \sin \left(\frac{5}{2}y \right) + \frac{4}{25} \cos \left(\frac{5}{2}y \right) \right) (+c)$ $\int x^2 dy = \frac{3}{8}y - \frac{1}{16}y^3 + \frac{1}{40}y \sin \left(\frac{5}{2}y \right) + \frac{1}{100} \cos \left(\frac{5}{2}y \right) (+c)$	A1	1.1b
$\int_{-1.545}^{1.257} x^2 dy = \frac{1}{16} \left[6y - y^3 + \frac{2}{5}y \sin \left(\frac{5}{2}y \right) + \frac{4}{25} \cos \left(\frac{5}{2}y \right) \right]_{-1.545}^{1.257}$ $= \frac{1}{16} (5.3954 \dots - (-6.1101 \dots)) = \dots$ $= (0.3372 \dots) - (-0.3818 \dots) = \dots$	M1	3.4
$\text{Volume} = \pi \times \frac{11.505 \dots}{16} = 2.26 \text{ cm}^3 \text{ (2.2591159...) cso}$	A1	3.2a
	(6)	

Notes

B1: Selects the correct volume of revolution formula to use, with correct limits in evidence, could appear later in their working

M1: Attempts to integrate with the correct form for the constant and term in y^2

M1: Applies integration by parts fully on the $y \cos \left(\frac{5}{2}y \right)$ term in the correct direction. Allow slips in the coefficients, but the form must be correct.

A1: Correct integration of the x^2 equation.

M1: Applies the correct limits to their integral provided there was some attempt at integration. No need for the π for this mark. **Condone working in degrees (5.74 ...) - (-5.38...), award this mark following correct integration.** If their integration is incorrect you will need to check the use of limits in radians only if they do not show the substitution.

A1: Correct volume including units. Accept awrt 2.26 cm^3 . Allow $0.719\pi \text{ cm}^3$. Correct solution only

Note: All previous marks must have been scored to award this final accuracy mark

Special case: Use of calculator for all the integration can score a maximum of **B1M0M0A0M1A0**

Mark scheme (b)

Scheme	Marks	AO
Max volume for 100 berries (as we know volume of the largest) is $100 \times 2.26 \simeq 226$	B1ft	1.1b
Reason e.g. not all the berries will become juice (e.g. skin, flesh, seeds may not pulp) or not all will be as big as the largest, $150 < 200$ or $300 > 200$ If their value	B1ft	2.2b

• is less than 220 – so not likely to produce 200 cm³ of juice. • is between 220 and 250 – they can conclude either way • is greater than 250 – so likely to produce 200 cm³ of juice.		
	(2)	
	(8 marks)	

Notes

(corrected from the printed mark scheme: the subtotal for (b) is printed as (1); part (b) carries 2 marks, B1ft B1ft)

B1ft: Attempts to estimate the volume of juice produced by 100 berries - look for their (a) multiplied by 100.

B1ft: Draws a suitable conclusion with reason given, see scheme

3.

Mark scheme (a)

Scheme	Marks
$\int \sin^2 \theta \, d\theta = \frac{1}{2} \int (1 - \cos 2\theta) \, d\theta = \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \quad (+C)$	M1 A1
	(2)

Mark scheme (b)

Scheme	Marks
$x = \tan \theta \Rightarrow \frac{dx}{d\theta} = \sec^2 \theta$	
$\pi \int y^2 \, dx = \pi \int y^2 \frac{dx}{d\theta} \, d\theta = \pi \int (2 \sin 2\theta)^2 \sec^2 \theta \, d\theta$	M1 A1
$= \pi \int \frac{(2 \times 2 \sin \theta \cos \theta)^2}{\cos^2 \theta} \, d\theta$	M1
$= 16\pi \int \sin^2 \theta \, d\theta \quad k = 16\pi$	A1
$x = 0 \Rightarrow \tan \theta = 0 \Rightarrow \theta = 0, \quad x = \frac{1}{\sqrt{3}} \Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$	B1
$\left(V = 16\pi \int_0^{\frac{\pi}{6}} \sin^2 \theta \, d\theta \right)$	
	(5)

Mark scheme (c)

Scheme	Marks
$V = 16\pi \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{6}}$	M1
$= 16\pi \left[\left(\frac{\pi}{12} - \frac{1}{4} \sin \frac{\pi}{3} \right) - (0 - 0) \right] \quad \text{Use of correct limits}$	M1
$= 16\pi \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} \right) = \frac{4}{3}\pi^2 - 2\pi\sqrt{3} \quad p = \frac{4}{3}, q = -2$	A1
	(3)

	(10 marks)
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