

QQQ – Core Pure 1 – Chapter 8 – Proof by Induction (Version 2)

Total Marks: 25

(25 = Platinum, 23 = Gold, 20 = Silver, 17 = Bronze)

1. The sequence $u_1, u_2, u_3, \dots$ is defined by

$$u_1 = 1$$

$$u_{n+1} = u_n + 3n$$

Prove by induction that for all integers $n \geq 1$

$$u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$$

[4 marks]

(Total 4 marks)

2. Astrid is solving this mathematics problem:

The series S_n is defined by

$$S_n = 2 + 4 + 6 + \dots + 2n \quad (n \in \mathbb{Z}, n \geq 1)$$

Prove by induction that

$$S_n = n(n + 1)$$

Astrid's solution is as follows:

Assume the result is true for $n = k$

Then

$$S_k = k(k + 1)$$

$$S_{k+1} = S_k + 2(k + 1)$$

$$S_{k+1} = k(k + 1) + 2(k + 1)$$

$$S_{k+1} = (k + 2)(k + 1)$$

$$S_{k+1} = (k + 1)((k + 1) + 1)$$

So the result is also true for $n = k + 1$

The result is true for $n = 1$

It is true for $n = k$, and also true for $n = k + 1$

Hence, by induction $S_n = n(n + 1)$ for all integers $n \geq 1$

- (a) (i) Chloe says that Astrid missed out an essential part of the proof, which could have been written at the start.

Explain what Astrid missed out.

[1 mark]

- (ii) Write down the working that Astrid missed out.

[1 mark]

- (b) (i) One statement in the last three lines of Astrid's solution is written incorrectly.

Which statement is written incorrectly?

Tick (✓) **one** box.

[1 mark]

- The result is true for $n = 1$

- It is true for $n = k$, and also true for $n = k + 1$
- Hence, by induction $S_n = n(n + 1)$ for all integers $n \geq 1$

(ii) Write out a correct statement which should replace the incorrect statement identified in part (b)(i) **[1 mark]**

(Total 4 marks)

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3. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$$

(5)

(Total 5 marks)

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4. Prove by induction that $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ for all positive integers n . **[6]**

(Total 6 marks)

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5. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(6)

(Total 6 marks)

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Solutions

1.

Scheme	Marks	AO
Shows that $u_1 = \frac{3}{2} \times 1^2 - \frac{3}{2} \times 1 + 1 = 1$	B1	1.1b
Assumes the result is true for $n = k$ and states $u_{k+1} = \frac{3}{2}k^2 - \frac{3}{2}k + 1 + 3k$ at any point	M1	3.1a
Completes correct working to deduce that $\frac{3}{2}k^2 - \frac{3}{2}k + 1 + 3k$ and $\frac{3}{2}(k+1)^2 - \frac{3}{2}(k+1) + 1$ are equivalent	B1	2.2a
Concludes a reasoned argument by stating: $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ true for $n = 1$ (seen anywhere) If $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ is true for $n = k$, then true for $n = k + 1$ Hence (by induction) $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ is true for all integers $n \geq 1$ Condone “the result/formula” instead of $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$	R1	2.1
	(4 marks)	

Typical solution

Let $n = 1$; then the formula gives

$$u_1 = \frac{3}{2} \times 1^2 - \frac{3}{2} \times 1 + 1 = 1$$

(So the formula is true for $n = 1$)

Assume the formula is true for $n = k$

Then we want to show that

$$\begin{aligned}u_{k+1} &= \frac{3}{2}(k+1)^2 - \frac{3}{2}(k+1) + 1 \\&= \frac{3}{2}(k^2 + 2k + 1 - k - 1) + 1 \\&= \frac{3}{2}(k^2 + k) + 1\end{aligned}$$

But

$$\begin{aligned}u_{k+1} &= \frac{3}{2}k^2 - \frac{3}{2}k + 1 + 3k \\&= \frac{3}{2}(k^2 + k) + 1\end{aligned}$$

as required

So the formula is also true for $n = k + 1$

The formula for u_n is true for $n = 1$

If the formula is true for $n = k$, then the formula is also true for $n = k + 1$

Hence by induction $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ is true for all integers $n \geq 1$

2.

Mark scheme (a)

Scheme	Marks	AO
(i) States that Astrid did not show that the result is true for $n = 1$	E1	2.3
	(1)	
(ii) Shows that the result is true for $n = 1$ for both definition and formula	E1	2.4
	(1)	

Typical solution

(i)

Astrid did not show that the result is true for $n = 1$

(ii)

$$S_1 = 2$$

and

$$S_1 = 1 \times (1 + 1) = 2$$

Mark scheme (b)

Scheme	Marks	AO
(i) Ticks 2nd answer.	E1	2.3
	(1)	
(ii) Writes a correct statement to replace the incorrect statement.	R1	2.1
	(1)	
	(4 marks)	

Typical solution

(i)

It is true for $n = k$, and also true for $n = k + 1$

(ii)

If true for $n = k$, it is also true for $n = k + 1$

3.

Scheme	Marks
When $n = 1$, LHS = $\frac{1}{1 \times 2} = \frac{1}{2}$, RHS = $\frac{1}{1 + 1} = \frac{1}{2}$. So LHS = RHS and result true for $n = 1$	B1
Assume true for $n = k$; $\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$ and so $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$	M1
$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$	M1 A1
and so result is true for $n = k + 1$ (and by induction true for $n \in \mathbf{Z}^+$)	B1
	[5]

Notes

Evaluate both sides for first B1

Final two terms on second line for first M1

Attempt to find common denominator for second M1.

Second M1 dependent upon first.

$$\frac{k+1}{k+2} \text{ for A1}$$

'Assume true for $n = k$ ' and 'so result true for $n = k + 1$ ' and correct solution for final B1

4.

Scheme	Marks	AO
When $n = 1$, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{pmatrix}$	B1	2.1
[Assume] $n = k$: $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$	M1	2.2a
Then $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$	M1	1.1
$= \begin{pmatrix} 1 & 1 + 2^{k+1} - 2 \\ 0 & 2^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix}$	A1*	1.1
so if true for $n = k$ then true for $n = k + 1$ [as true for $n = 1$] therefore true for all n .	M1dep* A1dep*	2.2a 2.4
	[6]	

Notes

M1: (2nd) or $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$

A1*: or $\begin{pmatrix} 1 & 2^k - 1 + 2^k \\ 0 & 2^{k+1} \end{pmatrix}$

M1dep*: must clearly 'assume' $n = k$

'true for $n = k$ and $k + 1$ ' is M0

A1dep*: must have established truth for $n = 1$ for final mark
but need not re-state it at the end

5.

Way 1: $f(k + 1) - f(k)$

Scheme	Marks	AO
When $n = 1$, $2^{n+2} + 3^{2n+1} = 2^3 + 3^3 = 35$ Shows the statement is true for $n = 1$, allow 5(7)	B1	2.2a
Assume true for $n = k$, so $2^{k+2} + 3^{2k+1}$ is divisible by 7	M1	2.4
$f(k + 1) - f(k) = 2^{k+3} + 3^{2k+3} - (2^{k+2} + 3^{2k+1})$	M1	2.1
$= 2 \times 2^{k+2} + 9 \times 3^{2k+1} - 2^{k+2} - 3^{2k+1}$ $= 2^{k+2} + 8 \times 3^{2k+1}$ $= f(k) + 7 \times 3^{2k+1}$ or $8f(k) - 7 \times 2^{k+2}$	A1	1.1b
$f(k + 1) = 2f(k) + 7 \times 3^{2k+1}$ or $9f(k) - 7 \times 2^{k+2}$	A1	1.1b
If true for $n = k$ then true for $n = k + 1$ and as it is true for $n = 1$ the statement is true for all (positive integers) n	A1	2.4
	(6)	
	(6 marks)	

Notes

Way 1: $f(k + 1) - f(k)$

B1: Shows that $f(1) = 35$ and concludes or shows divisible by 7. This may be seen in the final statement.

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k + 1) - f(k)$

A1: Achieves a correct expression for $f(k + 1) - f(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks, look out for scoring B1 in this statement. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Alternative: Way 2: $f(k + 1)$

Scheme	Marks	AO
When $n = 1$, $2^{n+2} + 3^{2n+1} = 2^3 + 3^3 = 35$ So the statement is true for $n = 1$	B1	2.2a

Assume true for $n = k$, so $2^{k+2} + 3^{2k+1}$ is divisible by 7	M1	2.4
$f(k+1) = 2^{(k+1)+2} + 3^{2(k+1)+1}$	M1	2.1
$f(k+1) = 2^{k+3} + 3^{2k+3} = 2 \times 2^{k+2} + 9 \times 3^{2k+1}$ $= 2(2^{k+2} + 3^{2k+1}) + 7 \times 3^{2k+1}$ $= 2f(k) + 7 \times 3^{2k+1}$ or $9f(k) - 7 \times 2^{k+2}$	A1 A1	1.1b 1.1b
If true for $n = k$ then true for $n = k + 1$ and as it is true for $n = 1$ the statement is true for all (positive integers) n	A1	2.4
	(6)	

B1: Shows that $f(1) = 35$ and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)$

A1: Correctly obtains either $2f(k)$ or $7 \times 3^{2k+1}$ or either $9f(k)$ or $-7 \times 2^{k+2}$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Alternative: Way 3: $f(k+1) - mf(k)$

Scheme	Marks	AO
When $n = 1$, $2^{n+2} + 3^{2n+1} = 2^3 + 3^3 = 35$ So the statement is true for $n = 1$	B1	2.2a
Assume true for $n = k$, so $2^{k+2} + 3^{2k+1}$ is divisible by 7	M1	2.4
$f(k+1) - mf(k) = 2^{k+3} + 3^{2k+3} - m(2^{k+2} + 3^{2k+1})$ $= 2 \times 2^{k+2} + 9 \times 3^{2k+1} - m \times 2^{k+2} - m \times 3^{2k+1}$ $= (2 - m)2^{k+2} + 9 \times 3^{2k+1} - m \times 3^{2k+1}$ $= (2 - m)(2^{k+2} + 3^{2k+1}) + 7 \times 3^{2k+1}$	M1 A1	2.1 1.1b
$f(k+1) = (2 - m)(2^{k+2} + 3^{2k+1}) + 7 \times 3^{2k+1} + mf(k)$	A1	1.1b
If true for $n = k$ then true for $n = k + 1$ and as it is true for $n = 1$ the statement is true for all (positive integers) n	A1	2.4
	(6)	

B1: Shows that $f(1) = 35$ and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k + 1) - mf(k)$

A1: Achieves a correct expression for $f(k + 1) - mf(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.