

QQQ – Core Pure 1 – Chapter 8 – Proof by Induction (Version 1)

Total Marks: 25

(25 = Platinum, 23 = Gold, 20 = Silver, 17 = Bronze)

1. Prove by induction that, for all integers $n \geq 1$,

$$\sum_{r=1}^n r^3 = \left\{ \frac{1}{2}n(n+1) \right\}^2$$

[4 marks]

(Total 4 marks)

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2. Prove by induction that, for all $n \in \mathbb{N}$, the expression

$$5^n - 2^n$$

is divisible by 3

[4 marks]

(Total 4 marks)

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3. Prove by induction that for $n \in \mathbb{N}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

(5)

(Total 5 marks)

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4. Prove that $3^n > 10n$ for all integers $n \geq 4$.

[5]

(Total 5 marks)

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5. A sequence can be described by the recurrence formula

$$u_{n+1} = 2u_n + 1, \quad n \geq 1, \quad u_1 = 1$$

(a) Find u_2 and u_3 . **(2)**

(b) Prove by induction that $u_n = 2^n - 1$ **(5)**

(Total 7 marks)

Solutions

1.

Scheme	Marks	AO
Demonstrates the result for $n = 1$ and states that it is true for $n = 1$	B1	2.2a
Assumes the result true for $n = k$ (PI by algebraic working) and adds $(k + 1)^3$ to $= \left(\frac{1}{2}k(k + 1) \right)^2$	M1	2.1
Obtains $= \frac{1}{4}(k + 1)^2(k + 2)^2$ from $= \left(\frac{1}{2}k(k + 1) \right)^2 + (k + 1)^3$	A1	1.1b
Concludes a reasoned argument by stating that $\sum_{r=1}^n r^3 = \left\{ \frac{1}{2}n(n + 1) \right\}^2$ is true for $n = 1$, and that $\sum_{r=1}^k r^3 = \left(\frac{1}{2}k(k + 1) \right)^2$ implies $\sum_{r=1}^{k+1} r^3 = \left(\frac{1}{2}(k + 1)(k + 2) \right)^2$ And hence, by induction, that $\sum_{r=1}^n r^3 = \left\{ \frac{1}{2}n(n + 1) \right\}^2$ is true for all integers $n \geq 1$	R1	2.1
	(4 marks)	

Typical solution

Let $n = 1$

Then $\left(\frac{1}{2}n(n+1)\right)^2 = 1^2 = 1$ and $\sum_{r=1}^n r^3 = 1$

So the result is true for $n = 1$

Assume the result is true for $n = k$

Then

$$\sum_{r=1}^k r^3 = \left(\frac{1}{2}k(k+1)\right)^2$$

and

$$\begin{aligned}\sum_{r=1}^{k+1} r^3 &= \left(\frac{1}{2}k(k+1)\right)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \left(\frac{1}{2}(k+1)(k+2)\right)^2\end{aligned}$$

The result is true for $n = 1$; if true for $n = k$, then it's also true for $n = k + 1$ and hence by induction

$$\sum_{r=1}^n r^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$$

for all integers $n \geq 1$

2.

Scheme	Marks	AO
Writes $5^1 - 2^1 = 3$ Accept $5^0 - 2^0 = 0$	B1	2.1
Assumes $5^k - 2^k$ is divisible by 3 and considers $5^{k+1} - 2^{k+1}$	M1	2.4
Completes rigorous working to show that $5^{k+1} - 2^{k+1}$ is divisible by 3	A1	2.2a

Concludes a reasoned argument by stating that $5^n - 2^n$ is divisible by 3 when $n = 1$ (or $n = 0$), and that if $5^n - 2^n$ is divisible by 3 when $n = k$ then it is also divisible by 3 when $n = k + 1$ and hence (by induction) $5^n - 2^n$ is divisible by 3 for all $n (\in \mathbf{N})$ Condone reference to 'rule' / 'statement' in their final statement. Must define their m to be an integer.	R1	2.1
	(4 marks)	

Typical solution

$$5^1 - 2^1 = 3$$

3 is divisible by 3

So the rule is true for $n = 1$

If the rule is true for $n = k$ then $5^k - 2^k = 3m$ for some integer m

$$\Rightarrow 5^k = 3m + 2^k$$

$$\begin{aligned}
 5^{k+1} - 2^{k+1} &= 5 \times 5^k - 2 \times 2^k \\
 &= 5(3m + 2^k) - 2 \times 2^k \\
 &= 15m + 5 \times 2^k - 2 \times 2^k \\
 &= 15m + 3 \times 2^k \\
 &= 3(5m + 2^k) \\
 &= 3 \times \text{integer}
 \end{aligned}$$

So the rule is also true for $n = k + 1$

Therefore, by induction

$5^n - 2^n$ is divisible by 3 for all $n \in \mathbf{N}$

3.

Scheme	Marks	AO
$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$		

$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & -2 \times 1 \\ 0 & 1 \end{pmatrix}$ (so true when $n = 1$)	B1	2.2a
(Assume true for $n = k$, then) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	M1	2.1
$= \begin{pmatrix} 1 & -2k - 2 \\ 0 & 1 \end{pmatrix}$ or $= \begin{pmatrix} 1 & -2 - 2k \\ 0 & 1 \end{pmatrix}$	A1	1.1b
$= \begin{pmatrix} 1 & -2(k+1) \\ 0 & 1 \end{pmatrix}$	A1	2.2a
Hence result is true for $n = k + 1$. As <u>true for $n = 1$</u> and have shown <u>if true for $n = k$ then it is true for $n = k + 1$</u> , so it is <u>true for all n</u> .	A1	2.5
	(5)	
	(5 marks)	

Notes

B1: Shows true for $n = 1$.

Need to see $n = 1$ **substituted** into rhs. The minimum for B1 would be $\begin{pmatrix} 1 & -2 \times 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

There is no need to state “True for $n = 1$ ”

M1: (Assumes for $n = k$ and) multiplies original matrix by k th power matrix either way round. Note that the assumption statement is not needed for this mark (but see below) so just look for:

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

A1: Achieves a correct unsimplified matrix.

A1: Reaches correct form for the matrix with the $k + 1$ factored out with no errors and the correct unsimplified matrix seen previously. Note that the result may be proved by equivalence (see below).

A1: Correct conclusion **with assumption made** (which may be implied in their conclusion if they say “if true for $n = k$ then...”). This mark is dependent on all the previous marks apart from the B mark and is gained by conveying all the underlined points.

Allow this mark to score as long as all the underlined points are seen as narrative in their solution.

There must be the assumption statement somewhere or the “if...then...” idea in the conclusion. If awarded for the assumption statement condone e.g. true for $n = k$ in the conclusion.

The conclusion must convey the “if true for $n = k$ then true for $n = k + 1$ ” idea and not e.g. true for k , true for $k + 1$, true for 1 therefore...but see the previous note.

For the “true for all n ” part condone e.g. “true for n ”, “true for all integers after 1”, “true for $\mathbb{Z}^+$ ”,

But do **not** allow “true for all values”, “true for all real numbers”

Q4 Extra Notes:

1. For candidates who use n instead of k throughout withhold the final mark if the work is otherwise correct.

2. For equivalence proofs, this would be minimally acceptable:

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \times 1 \\ 0 & 1 \end{pmatrix} \text{ B1}$$

“Need to prove” oe e.g. “target is”, “ $n = k + 1$ ”, “need” etc. $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -2k - 2 \\ 0 & 1 \end{pmatrix}$

Note that there must be a reference to $k + 1$ as above or e.g. (Target =)

$$\begin{pmatrix} 1 & -2(k+1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2k - 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2k - 2 \\ 0 & 1 \end{pmatrix} \text{ M1A1A1}$$

Hence result is true for $n = k + 1$. As true for $n = 1$ and have shown if true for $n = k$ then it is true for $n = k + 1$, so it is true for all n . **A1**

Without the “Need to prove...” oe the response would score B1M1A1 and then A1 if the factorised form was shown or equivalence shown and then A1 for the correct conclusion.

3. Allow e.g. “correct” for “true” in the conclusion

4.

Scheme	Marks	A0
If $n = 4$, LHS = $3^4 = 81$ RHS = $10 \times 4 = 40 < 81 = \text{LHS}$ So true for $n = 4$	B1	2.5
(Assume that) $3^k > 10k$ (for some integer $k \geq 4$).	M1	2.1
$3^{k+1} = 3 \times 3^k > 3 \times 10k \dots$	M1	1.1

$\dots = 30k = 10(k + 1) + 10(2k - 1) > 10(k + 1)$ since $2k - 1 > 0$ since $k \geq 4$ i.e. if $3^k > 10k$ then $3^{k+1} > 10(k + 1)$	A1	2.2a
So true for $n = k \Rightarrow$ true for $n = k + 1$. But true for $n = 4$. So true for all integers $n \geq 4$	A1	2.4
	[5]	

Notes

B1: Basis case. Comparison must be explicit and correct

An assertion without calculation such as e.g. $3^4 > 10 \times 4$ is insufficient for **B1**.

BOD statements such as “therefore true when $n = 1$ ”

M1: (1st) Inductive hypothesis set up

M1: (2nd) Considering for $k + 1$ and using inductive hypothesis correctly

Asserting $3^{k+1} > 10(k + 1)$ without justification gets **M0**.

Could compare $3k$ and $k + 1$

A1: (1st) Showing enough working to establish statement for $k + 1$. Must be justified but justification could be e.g. $k > 1$.

A1: (2nd) Clear and complete conclusion, following a **correct and complete** proof with no incorrect statements. **Must** be $n \geq 4$ not eg 0 or 1 for **A1**.

This mark must only be awarded if the language and notation in the whole proof and conclusion is correct.

5.

Mark scheme (a)

Scheme	Marks
$u_2 = 3, u_3 = 7$	B1, B1
	(2)

Mark scheme (b)

Scheme	Marks
At $n = 1, u_1 = 2^1 - 1 = 1$ and so result true for $n = 1$	B1

Assume true for $n = k$; $u_k = 2^k - 1$	
and so $u_{k+1} (= 2u_k + 1) = 2(2^k - 1) + 1$ <i>Substitutes u_k into u_{k+1} (must see this line)</i> <i>Correct expression</i>	M1 A1
$u_{k+1} (= 2^{k+1} - 2 + 1) = 2^{k+1} - 1$ <i>Correct completion to $u_{k+1} = 2^{k+1} - 1$</i>	A1
Must see 4 things: <u>true for $n = 1$</u> , <u>assumption true for $n = k$</u> , <u>said true for $n = k + 1$</u> and therefore <u>true for all n</u> <i>Fully complete proof with no errors and comment. All the previous marks in (b) must have been scored.</i>	Alcso
	(5)
	(7 marks)

Notes

Ignore any subsequent attempts e.g. $u_{k+2} = 2u_{k+1} + 1 = 2(2^{k+1} - 1) + 1$ etc.