

3A Continuous Random Variables

Continuous random variable:

For a continuous random variable X , $P(X = x) =$ for any single value x , because

Probability density function (p.d.f.):

If X is a continuous random variable with p.d.f. $f(x)$, then:

- $f(x)$ for all $x \in \mathbb{R}$
- $P(a < X < b) =$
- the total area under the p.d.f. is

1. In each case, state whether the function could be a probability density function, where k is a positive constant.

a
$$f(x) = \begin{cases} 2x + 1 & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

b
$$f(x) = \begin{cases} k(2 - x) & -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

c
$$f(x) = \begin{cases} kx(5 - x) & 0 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

2. The random variable X has probability density function

$$f(x) = \begin{cases} kx(4-x) & 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find:

- a) the value of k and sketch the p.d.f.

- b) $P(2.5 < X < 3)$, giving your answer to 3 decimal places.

3. The random variable X has probability density function

$$f(x) = \begin{cases} k & 1 < x < 2 \\ k(x-1) & 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

- a) Sketch $f(x)$.

b) Find the value of k .

c) Find $P(X > 3)$.

3B The Cumulative Distribution Function

Cumulative distribution function (c.d.f.), $F(x)$:

$$F(x) = P(X \leq x) =$$

In terms of area under the p.d.f., $F(x)$ is

Going between the p.d.f. and the c.d.f.:

$$f(x) = \qquad \qquad \qquad F(x) =$$

To specify $F(x)$ fully you must

1. The random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{4}x & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find $F(x)$.

2. The random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{5} & 1 < x < 2 \\ \frac{1}{5}(x - 1) & 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Specify fully the cumulative distribution function of X .

3. The random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{5}x + \frac{3}{20}x^2 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

Find:

a) $P(X \leq 1.5)$

b) $P(0.5 \leq X \leq 1.5)$

c) $P(X = 1)$

d) the probability density function, $f(x)$.

3C Mean and Variance of a Continuous Distribution

If X is a continuous random variable with p.d.f. $f(x)$:

Mean or expected value: $E(X) = \mu =$

Variance: $\text{Var}(X) = \sigma^2 =$

Expected value of a function of X : $E(g(X)) =$

A convenient way to calculate the variance: $\text{Var}(X) =$

$E(aX + b) =$

$\text{Var}(aX + b) =$

If the p.d.f. is symmetric about $x = a$, then

1. A random variable Y has probability density function

$$f(y) = \begin{cases} \frac{1}{4}y & 1 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a) $E(Y)$

b) $\text{Var}(Y)$

c) $E(2Y - 3)$

d) $\text{Var}(2Y - 3)$

2. A random variable X has probability density function

$$f(x) = \begin{cases} \frac{3}{32}(3 + 2x - x^2) & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

- a) Sketch the probability density function.

b) Find $E(X)$.

3. The random variable X has probability density function

$$f(x) = \begin{cases} \frac{2}{15}x & 0 \leq x < 3 \\ \frac{1}{5}(5 - x) & 3 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a) $E(X)$

b) $\text{Var}(X)$

4. In a block-building video game, a cube of side length X pixels is randomly generated. X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{800}(x + 10) & 20 \leq x \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

a) Find:

i) $E(X)$

ii) $E(X^2)$

iii) $\text{Var}(X)$

b) Find the expected volume of the cube.

3D Mode, Median, Percentiles and Skewness

The mode of a continuous random variable is

How to find the mode:

1. The random variables X and Y have probability density functions f(x) and g(y) respectively.

$$f(x) = \begin{cases} 12x^2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
$$g(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mode of:

a) X

b) Y

If X is a continuous random variable with c.d.f. F(x):
the median m satisfies

the lower quartile Q_1 satisfies

the upper quartile Q_3 satisfies

the nth percentile P_n satisfies

2. A continuous random variable X has probability density function

$$f(x) = \begin{cases} 4x - 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find:

- a) the c.d.f. of X

- b) the median value of X.

3. A continuous random variable X has the cumulative distribution function

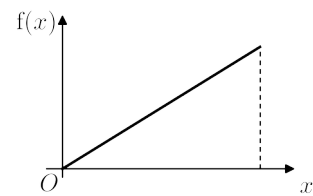
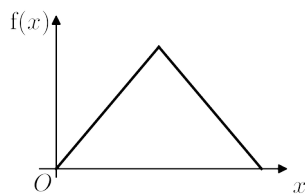
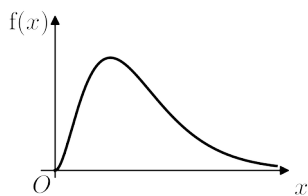
$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{5}x - \frac{1}{5} & 1 \leq x \leq 2 \\ \frac{x^2}{10} - \frac{x}{5} + \frac{1}{5} & 2 < x < 4 \\ 1 & x \geq 4 \end{cases}$$

Find:

- a) the interquartile range

b) the 10th percentile.

Skewness: sketch and label each p.d.f.



Positive skew: < <

Negative skew: < <

4. A continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{10}x & 0 \leq x < 2 \\ \frac{1}{4} - \frac{1}{40}x & 2 \leq x < 10 \\ 0 & \text{otherwise} \end{cases}$$

a) Find:

i) the mean of X

ii) the mode of X .

b) Comment on the skewness of the distribution.

3E The Continuous Uniform Distribution

A continuous uniform distribution over the interval $[a, b]$ has p.d.f.

$$f(x) =$$

Notation:

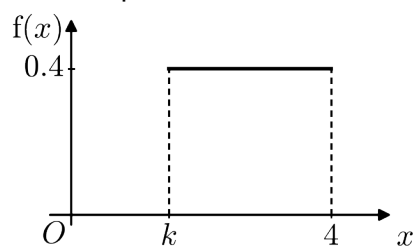
Sketch of the p.d.f.:

1. The continuous random variable X is uniformly distributed on $[3, 5]$. Find:

a) $P(3.2 < X < 4.3)$

b) k such that $P(2X < k - X) = 0.2$.

2. The continuous random variable X has p.d.f. as shown in the diagram.



Find:

- a) the value of k

b) $P(3 < X < 3.5)$

c) $P(X > 3 \mid X > 2)$

3. The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{5} & 3 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases}$$

a) Write down the name of this distribution.

The continuous random variable $Y = 12 - 3X$. Find:

b) $E(Y)$

c) $P(Y > 0)$

d) $P(X < 7 \mid Y < 0)$

4. The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Find:

a) $E(X)$

b) $\text{Var}(X)$

c) $F(x)$

For a continuous uniform distribution $U[a, b]$:

$E(X) =$

$\text{Var}(X) =$

$F(x) =$

5. The continuous random variable X is uniformly distributed over the interval $[4, 7]$. Find:
- a) $E(X)$
 - b) $\text{Var}(X)$
 - c) the cumulative distribution function of X , for all x .
6. The continuous random variable Y is uniformly distributed over the interval $[a, b]$. Given that $E(Y) = 1$ and $\text{Var}(Y) = 16/3$, find the value of a and the value of b .
7. The continuous random variable X is uniformly distributed over the interval $[-3, 5]$.
- a) Write down $E(X)$.
 - b) Use integration to find the variance of X .

3F Modelling with the Continuous Uniform Distribution

Real-life situations that can be modelled by a continuous uniform distribution:

Remember: $E(X^2)$ is not the same as $(E(X))^2$. $E(X^2) =$

1. The trunk of a small tree varies in diameter from 10 cm at the bottom to 2 cm at the top. The tree is cut horizontally at a randomly chosen point, and the radius R cm of the cross-section is modelled as $R \sim U[1, 5]$.

Find the expected value of the area, A , of the cross-section of the tree.

2. The length of a pencil is measured to the nearest cm. Write down the distribution of the rounding errors R .

3. Write down the name of the distribution you would recommend as a suitable model for each of the following situations.

a) The masses of 200 g tins of tomatoes produced on a production line.

b) The difference between the true length and the length of metal rods measured to the nearest centimetre.