

7A Probability Generating Functions

A probability generating function (p.g.f.) is:

A p.g.f. can only be used with:

If a discrete random variable X has probability mass function $P(X = x)$, then the probability generating function of X is

$$G_X(t) =$$

where t is:

For example, find the p.g.f. of X :

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.3	0.2

$$G_X(t) =$$

The coefficient of t^x is:

For any probability generating function, $G_X(1) =$ because

The p.g.f. in terms of expectation (given in the formulae booklet):

$$G_X(t) =$$

1. X is the discrete random variable that denotes the absolute difference of the scores when two fair dice are thrown. Construct the probability distribution of X and write down the probability generating function.

Complete the sample space diagram:

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0				
3	2		0			
4				0		
5					0	
6						0

x	0	1	2	3	4	5
$P(X = x)$						

2. The probability generating function of the discrete random variable X is given by $G_X(t) = k(1 + t)^2$
- a) Find the value of k .

- b) Write down the probability distribution of X .

x			
$P(X = x)$			

7B Probability Generating Functions of Standard Distributions

The probability generating functions of the standard distributions (given in the formulae booklet):

If $X \sim B(n, p)$: $G_X(t) =$

If $X \sim \text{Po}(\lambda)$: $G_X(t) =$

If $X \sim \text{Geo}(p)$: $G_X(t) =$

If $X \sim \text{Negative B}(r, p)$: $G_X(t) =$

'Write down' or 'find' the p.g.f. of a standard distribution means:

'Show from first principles' means:

Useful results: Maclaurin series $e^x =$

Sum to infinity of a geometric series $S_\infty =$

3. An archer hits the bullseye with probability 0.6. She fires three shots at a target. The random variable X represents the number of times she hits the bullseye. Find the probability generating function of X .

x	0	1	2	3
$P(X = x)$				

4. X is a discrete random variable such that $X \sim \text{Po}(1.1)$. Show, from first principles, that the probability generating function of X is given by $G_X(t) = e^{1.1(t-1)}$

5. A fair tetrahedral dice is rolled until the dice shows a four. The discrete random variable X represents the number of rolls up to and including the first instance of a four. Show, from first principles, that the probability generating function for X is given by

$$G_X(t) = \frac{\frac{1}{4}t}{1 - \frac{3}{4}t}$$

6. The discrete random variable $X \sim \text{Negative B}(r, p)$. Prove, from first principles, that the probability generating function of X can be written as

$$G_X(t) = \left(\frac{pt}{1 - (1 - p)t} \right)^r$$

You may quote the following result without proof:

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1} q^{x-r} = (1-q)^{-r} \quad \text{where } q = 1 - p$$

7C Mean and Variance of a Distribution

If X is a discrete random variable with probability generating function $G_X(t)$:

$$E(X) =$$

$$\text{Var}(X) =$$

Proof of the result for $E(X)$: $G_X(t) = E(t^X) = \sum t^x P(X = x)$

Proof of the result for $\text{Var}(X)$:

Differentiating term by term: if $G_X(t) = P(X = 0) + P(X = 1)t + P(X = 2)t^2 + P(X = 3)t^3 + \dots$

then $G'_X(t) =$

and $G''_X(t) =$

So $G_X(0) =$ $G'_X(0) =$ $G''_X(0) =$ and in general $P(X = n) =$

1. The discrete random variable X has a probability generating function given by

$$G_X(t) = \frac{1}{100\,000}(9 + t^2)^5$$

Find:

a) $E(X)$

b) $\text{Var}(X)$

2. A discrete random variable X has a probability generating function given by $G_X(t) = a + bt + ct^2$, where a , b and c are constants. Given that the expected value of X is $7/6$ and the variance of X is $29/36$, find the values of a , b and c .

7D Sums of Independent Random Variables

Dice A has three faces showing 1 and three faces showing 2. Dice B has two faces showing 0 and four faces showing 1. X and Y are the scores on dice A and dice B.

x	1	2
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

y	0	1
$P(Y = y)$	$\frac{1}{3}$	$\frac{2}{3}$

Write down $G_X(t)$ and $G_Y(t)$:

Complete the distribution of $Z = X + Y$ and write down $G_Z(t)$:

z	1	2	3
$P(Z = z)$			

Find $G_X(t) \times G_Y(t)$. What do you notice?

If X and Y are independent random variables with p.g.f.s $G_X(t)$ and $G_Y(t)$, the p.g.f. of $Z = X + Y$ is given by

$$G_Z(t) =$$

1. Two independent discrete random variables X and Y have probability generating functions

$$G_X(t) = \frac{1}{2} + \frac{1}{2}t \quad \text{and} \quad G_Y(t) = \frac{1}{3} + \frac{1}{2}t + \frac{1}{6}t^2$$

- a) Find the probability generating function of the random variable $Z = X + Y$.

- b) Use probability generating functions to show that $E(Z) = E(X) + E(Y)$.

2. A random variable X has a probability generating function $G_X(t) = \frac{2}{3} + \frac{1}{3}t$

a) Write down the probability distribution of X .

x		
$P(X = x)$		

b) $Y = 2X + 1$. Write down the probability distribution of Y and hence find the probability generating function of Y .

y		
$P(Y = y)$		

c) Verify that $G_Y(t) = tG_X(t^2)$.

Linear transformations: if X has p.g.f. $G_X(t)$, then the p.g.f. of $Y = aX + b$ (where a and b are positive integers) is

$$G_Y(t) =$$

This result is not in the formulae booklet, so