

2A/B The Poisson Distribution

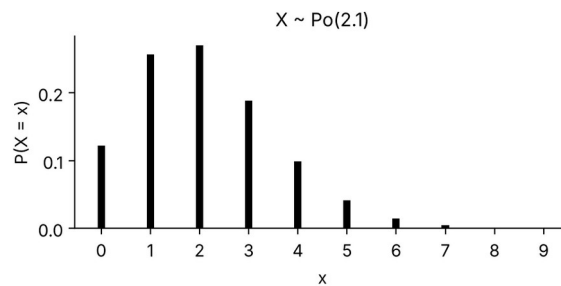
The Poisson distribution is used to model:

If X has a Poisson distribution with parameter λ , we write:

Probability function:

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, 3, \dots$$

The values that X can take:



Why the probabilities add up to 1 (use the series expansion below):

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots + \frac{\lambda^r}{r!} + \dots$$

1. The random variable $X \sim \text{Po}(2.1)$. Find:

a) $P(X = 3)$

b) $P(X \geq 1)$

c) $P(1 < X \leq 4)$

	2	3	4
P(X = x)			

Cumulative Poisson tables give:

Writing probabilities in terms of $P(X \leq x)$ so you can use the tables:

	In terms of $P(X \leq x)$
$P(X < a)$	
$P(X \geq a)$	
$P(X > a)$	
$P(a \leq X \leq b)$	

2. The random variable $X \sim \text{Po}(5)$. Find, using tables:

- a) $P(X \leq 3)$
- b) $P(X \geq 2)$
- c) $P(1 \leq X \leq 4)$

Extract from the cumulative Poisson tables, $P(X \leq x)$:

	$\lambda = 4.5$	$\lambda = 5.0$
x = 0	0.0111	0.0067
1	0.0611	0.0404
2	0.1736	0.1247
3	0.3423	0.2650
4	0.5321	0.4405

3. The random variable $X \sim \text{Po}(7.5)$. Find the values of a, b and c such that:

a) $P(X \leq a) = 0.2414$

b) $P(X < b) = 0.5246$

c) $P(X \geq c) = 0.3380$

Extract from the cumulative Poisson tables, $\lambda = 7.5$:

	4	5	6	7	8	9
P(X ≤ x)	0.1321	0.2414	0.3782	0.5246	0.6620	0.7764

2C Modelling with the Poisson Distribution

The Poisson distribution models the number of times, X , that:

For a Poisson distribution to be a good model, events must occur:

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The parameter λ represents:

Changing the length of the interval:

Examples of situations that could be modelled by a Poisson distribution:

1. An internet service provider has a large number of users regularly connecting to the internet. On average, 4 users every hour fail to connect to the internet at their first attempt.
 - a) Give two reasons why a Poisson distribution might be a suitable model for the number of failed connections every hour.
 - b) Find the probability that in a randomly chosen hour:
 - i) 2 users fail to connect at their first attempt
 - ii) more than 6 users fail to connect at their first attempt.
 - c) Find the probability that in a randomly chosen 90-minute period:
 - i) 5 users fail to connect at their first attempt
 - ii) fewer than 7 users fail to connect at their first attempt.

2D Adding Poisson Distributions

If $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Po}(\mu)$ are independent, then:

Watch out: before you add the parameters, the two random variables must:

1. If $X \sim \text{Po}(3.6)$ and $Y \sim \text{Po}(4.4)$, find:

a) $P(X + Y = 7)$

b) $P(X + Y \leq 5)$

2. The number of cars passing an observation point in a 5-minute interval is modelled by a Poisson distribution with mean 2. The number of other vehicles passing the observation point in a 15-minute interval is modelled by a Poisson distribution with mean 3. Find the probability that:

- a) exactly 5 vehicles, of any type, pass the observation point in a 10-minute interval

	5-minute interval	10-minute interval	15-minute interval
Cars, X	Po(2)		
Other vehicles, Y			Po(3)
All vehicles, X + Y			

- b) more than 8 vehicles, of any type, pass the observation point in a 15-minute interval.

2E Mean and Variance of a Poisson Distribution

If $X \sim \text{Po}(\lambda)$, then:

$$E(X) =$$

$$\text{Var}(X) =$$

Why this property is useful:

Calculating the mean and variance from summary data (Σx and Σx^2) for n observations:

$$\bar{x} =$$

$$\sigma^2 =$$

1. A botanist counts the number of daisies, x , in each of 80 randomly selected squares within a field. The results are summarised below.

$$\Sigma x = 295, \quad \Sigma x^2 = 1386$$

- a) Calculate the mean and the variance of the number of daisies per square for the 80 squares. Give your answers to 2 decimal places.

- b) Explain how the answers from part a support the choice of a Poisson distribution as a model.

- c) Using a suitable value for λ , estimate the probability that exactly 3 daisies will be found in a randomly selected square.

2F Mean and Variance of the Binomial Distribution

Reminder: the probability function for $X \sim B(n, p)$ is

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n - x}, \quad x = 0, 1, 2, \dots, n$$

If $X \sim B(n, p)$, then:

$$E(X) = \mu =$$

$$\text{Var}(X) = \sigma^2 =$$

1. A fair, five-sided spinner is spun 20 times. The random variable X represents the number of 5s obtained.

a) Find the mean and variance of X .

b) Find $P(X < \mu - \sigma)$.

2. A company produces a certain type of delicate component. The probability of any one component being defective is p . The probability of obtaining at least one defective component in a sample of 4 is 0.3439. The company produces 600 components in a day. Find the mean and variance of the number of defective components produced per day.

2G Using the Poisson Distribution to Approximate the Binomial Distribution

If $X \sim B(n, p)$, then X can be approximated by a Poisson distribution when:

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The approximating distribution is $X \approx \text{Po}(\quad)$, where $\lambda =$

Why the approximation works (compare the mean and variance of $B(n, p)$):

1. The random variable $X \sim B(200, 0.03)$.
 - a) Find $P(X = 4)$.

A Poisson variable $Y \sim \text{Po}(\lambda)$ is used to approximate X .

- b) Write down the value of λ and justify the use of a Poisson approximation in this context.

- c) Find $P(Y = 4)$ and comment on the accuracy of the approximation.

	Exact: $X \sim B(200, 0.03)$	Approximation: $Y \sim \text{Po}(\quad)$
$P(\dots = 4)$		

2. The probability of a component produced by a certain machine being faulty is 0.007. The number of faulty components in a batch of 1000 components is noted.
 - a) Find the probability that exactly 6 components are faulty.

b) Use a Poisson approximation to find the probability that more than 7 components are faulty.

c) Explain why the approximation in part b is valid.