

6A Reduction Formulae

A reduction formula is:

The formula for integration by parts:

How you use a reduction formula to find an integral:

1. Given that $I_n = \int x^n e^x dx$, where n is a positive integer,
a) show that $I_n = x^n e^x - n I_{n-1}$, $n \geq 1$

b) Hence find $\int x^4 e^x dx$

2.

a) Show that, if $I_n = \int \sin^n x dx$, then $I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$

b) Apply the reduction formula to find $\int \sin^4 x \, dx$

Why you may need an identity after integrating by parts:

3.

Show that, if $I_n = \int_0^1 x^n \sqrt{1-x} \, dx$, then $I_n = \frac{2n}{2n+3} I_{n-1}$, $n \geq 1$

4. $I_n = \int \tan^n x \, dx$, where n is a positive integer. By writing $\tan^n x$ as $\tan^{n-2} x \tan^2 x$, and using $1 + \tan^2 x \equiv \sec^2 x$, establish the reduction formula

$$I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}, \quad n \geq 2$$

5.

Consider the general definite integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

a) Prove that, for $n \geq 2$, $nI_n = (n-1)I_{n-2}$

b) Find the values of

$$\text{i) } \int_0^{\frac{\pi}{2}} \sin^5 x \, dx \qquad \text{ii) } \int_0^{\frac{\pi}{2}} \sin^6 x \, dx$$

6.

Given $I_n = \int_0^{\ln 2} \tanh^n x \, dx$, $n \geq 1$

a) Show that $I_n = I_{n-2} - \frac{1}{n-1} \left(\frac{3}{5}\right)^{n-1}$, $n \geq 2$

b) Hence, show that $\sum_{r=1}^{\infty} \frac{1}{2^r} \left(\frac{3}{5}\right)^{2r} = \ln\left(\frac{5}{4}\right)$

You may assume that the limit of I_n as $n \rightarrow \infty$ is 0.

7.

A general indefinite integral is defined as $I_n = \int \frac{\sin nx}{\sin x} dx, \quad n \geq 1$

a) By considering $I_{n+2} - I_n$, show that $I_{n+2} = \frac{2\sin((n+1)x)}{n+1} + I_n$

b) Hence, show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} dx = \frac{1}{15}(12\sqrt{3} - 17\sqrt{2})$

6B Arc Length

Arc length means:

Using Pythagoras on a small arc δs :

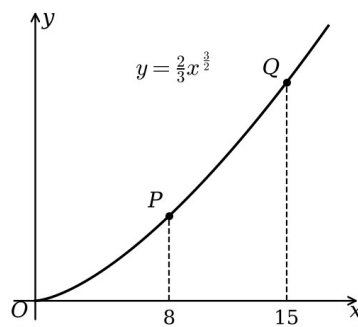
Arc length for $y = f(x)$, integrating with respect to x :

Arc length for $y = f(x)$, integrating with respect to y :

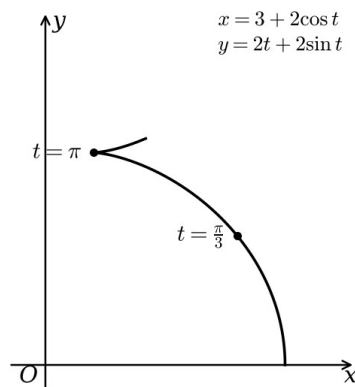
Arc length for a curve given parametrically, $x = f(t)$, $y = g(t)$:

Arc length for a curve with polar equation $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$:

1. Find the length of the arc PQ on the curve with equation $y = \frac{2}{3}x^{3/2}$, where the x-coordinates of P and Q are 8 and 15 respectively.



2. A curve is defined by the parametric equations $x = 3 + 2\cos t$, $y = 2t + 2\sin t$, $t \in \mathbb{R}$.

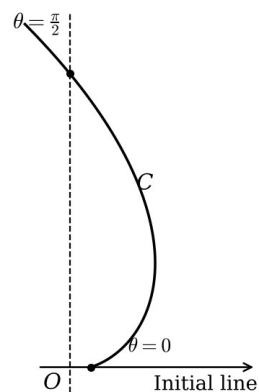


a)

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 16\cos^2 \frac{t}{2}$

- b) Find the exact length of this curve between the points where $t = \pi/3$ and $t = \pi$.

3. The curve C shown in the diagram has polar equation $r = 5e^{2\theta}$. Find the length of the arc of C between the points on the curve with $\theta = 0$ and $\theta = \pi/2$.



A useful substitution for $\int \sqrt{1+x^2} \, dx$:

4. Find the exact length of the arc on the parabola with equation $y = \frac{1}{2}x^2$, from the origin to the point $P(4, 8)$.

6C Area of a Surface of Revolution

A surface of revolution is:

Area of the thin band made by rotating a small arc δs , a distance y from the x -axis:

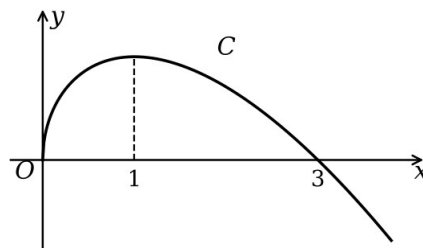
Surface area for $y = f(x)$ rotated through 2π about the x -axis:

Surface area for a Cartesian curve rotated through 2π about the y -axis (two forms):

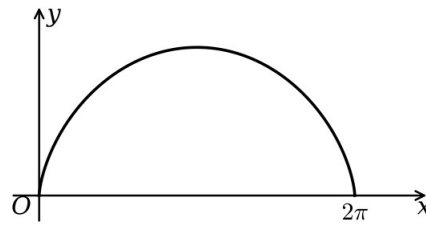
Surface area for a parametric curve $x = f(t)$, $y = g(t)$, rotated about the x -axis and about the y -axis:

Surface area for a polar curve $r = f(\theta)$, rotated about the initial line and about the line $\theta = \pi/2$:

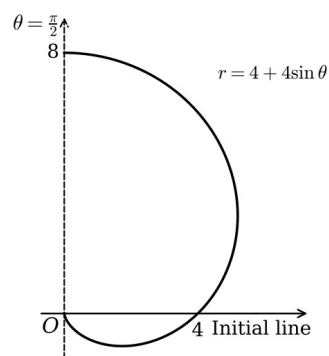
1. The curve C has equation $y = \frac{1}{3}\sqrt{x}(3 - x)$. The arc of the curve between the points with x -coordinates 1 and 3 is completely rotated about the x -axis. Find the area of the surface generated.



2. The curve with parametric equations $x = t - \sin t$, $y = 1 - \cos t$, from $t = 0$ to $t = 2\pi$, is rotated through 360° about the x-axis. Find the area of the surface generated.



3. The graph of the polar equation $r = 4 + 4\sin \theta$ from $\theta = -\pi/2$ to $\theta = \pi/2$ is shown below. The curve is rotated through 2π radians about the vertical line $\theta = \pi/2$, forming a surface of revolution. Find the exact area of the surface.



4. The arc of the curve with equation $y = \cosh x$, from the point $(0, 1)$ to $(\ln 2, 5/4)$, is rotated completely about the y -axis. Find the area of the surface generated.

