

5A/B Eigenvalues and Eigenvectors

An eigenvector of a matrix A is:

The eigenvalue corresponding to the eigenvector x is:

Under the transformation represented by A, what happens to an eigenvector? What does the eigenvalue tell you?

Eigenvectors and invariant lines:

If the corresponding eigenvalue is 1:

Why $\det(A - \lambda I) = 0$ gives the eigenvalues:

The characteristic equation of A:

1. Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$$

A normalised eigenvector:

If $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is an eigenvector, then $\hat{\mathbf{a}} =$

Normalised eigenvectors for question 1:

Repeated eigenvalues and complex eigenvalues:

Linearly independent vectors:

2. Find the eigenvalues and corresponding eigenvectors for these matrices:

a)

$$\mathbf{A} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

b)

$$\mathbf{B} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

c)

$$\mathbf{C} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

3. A transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$$

a) Find the eigenvalues of $\mathbf{A}$.

b) Find Cartesian equations of the two lines passing through the origin which are invariant under T .

Eigenvalues and eigenvectors of a 3×3 matrix (what kind of equation? how do you factorise it?):

Why a 3×3 matrix always has at least one real eigenvalue:

4. Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -4 & -3 \end{pmatrix}$$

5. The matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix}$$

- a) Show that -2 is the only real eigenvalue of $\mathbf{A}$.

- b) Find a normalised eigenvector of A corresponding to the eigenvalue -2 .

6. The matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

- a) Show that 1 is a repeated eigenvalue of A and find the other distinct eigenvalue.

- b) Find two linearly independent eigenvectors corresponding to the eigenvalue 1 .

5C Reducing Matrices to Diagonal Form

A diagonal matrix:

The leading diagonal:

The general 2×2 diagonal matrix:

To reduce a given matrix A to diagonal form:

- 1.
- 2.
- 3.
- 4.

1. For the matrix below:

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

a) Find a matrix P such that $D = P^{-1}AP$ is diagonal.

b) Write down the diagonal matrix D .

The elements on the leading diagonal of D :

A symmetric matrix:

An orthogonal matrix:

The procedure for orthogonal diagonalisation of a symmetric matrix A:

1.

2.

3.

4.

2. Reduce the matrix A to a diagonal matrix.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

3. Find a matrix P and a diagonal matrix D such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 4 & 2 & -3 \\ 4 & 2 & 3 \end{pmatrix}$$

4. The matrix

$$\mathbf{A} = \begin{pmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{pmatrix}$$

a) Verify that $(2, 1, 1)^T$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

b) Show that -6 is another eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

- c) Given that the third eigenvector of A is $(1, -1, -1)^T$, find a matrix P and a diagonal matrix D such that $P^TAP = D$.

5. The two sequences x_n and y_n satisfy the recurrence relations

$$x_{n+1} = 4x_n + y_n, \quad x_1 = 1 \quad y_{n+1} = 2x_n + 3y_n, \quad y_1 = 2 \quad n \geq 1$$

These recurrence relations can be written in matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{where } \mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

- a) Find the eigenvalues and corresponding eigenvectors of A.

- b) Hence write down matrices P and D such that $P^{-1}AP = D$ where D is a diagonal matrix.

New sequences u_n and v_n can be formed from x_n and y_n using the transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

- c)

$$\text{Show that } \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

d) Hence find closed form expressions for the original sequences x_n and y_n .

Powers of a diagonal matrix:

$$\mathbf{D} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Rightarrow \mathbf{D}^k =$$

If $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, then $\mathbf{A}^k =$

6. For the matrix below:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

a) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

b) Hence find $\mathbf{A}^6$.

5D The Cayley-Hamilton Theorem

The Cayley-Hamilton theorem:

If the characteristic equation of M is $\lambda^2 + a\lambda + b = 0$, then:

1. Demonstrate that the matrix A satisfies its own characteristic equation.

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix}$$

Using the Cayley-Hamilton theorem to find higher powers of a matrix:

2. Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix}$$

- a) find the characteristic equation of A .

- b) Hence show that $A^3 = 13A - 18I$.

Using the Cayley-Hamilton theorem to find the inverse of a matrix:

3. Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

a) find the characteristic equation of $\mathbf{M}$.

b) Hence, by using the Cayley-Hamilton theorem, find $\mathbf{M}^{-1}$.