4A Forming Recurrence Relations

A recurrence relation:

An initial condition:

First-order and second-order recurrence relations:

1. You owe £500 on a credit card that charges 1.5% interest each month. You decide to make no new charges and you pay off £50 each month. Formulate a recurrence relation that describes the balance remaining on the credit card after n months.

2. The deer population of a county was observed to be 1200 in a given year. The population is modelled to increase at a rate of 15% each year. Let d_n be the population of deer n years later. Explain why the deer population is modelled by the recurrence relation $d_n = 1.15d_{n-1}$, with $d_0 = 1200$.

3. A population of bacteria has initial size 200. After one hour, the population has reached 220. The population grows in such a way that the rate of growth doubles each hour. Write a recurrence relation to describe the number of bacteria, b_n, after n hours.

How to verify that a general term $u_n = f(n)$ satisfies a recurrence relation:

4. A sequence has the general term $u_n = 3n - 1$. Verify that the sequence satisfies the recurrence relation $u_n = 3 + u_{n-1}$.

5. A sequence has the general term $u_n = 2 \times 3^{n-1}$. Verify that the sequence satisfies the recurrence relation $u_n = 3u_{n-1}$.

Recursive form and closed form of a sequence:

4B Solving First-Order Recurrence Relations

The order of a recurrence relation:

A first-order linear recurrence relation can be written in the form:

It is homogeneous when:

Back substitution:

1. Find a closed form for the sequence $a_n = 5a_{n-1}$, $n > 0$; $a_0 = 1$.

The solution to the first-order homogeneous recurrence relation $u_n = au_{n-1}$ is:

2. Solve the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$, with $a_0 = 3$.

General solution and particular solution:

3. Solve the recurrence relation $a_n = -3a_{n-1}$, $n \geq 1$, with $a_1 = 6$.

4. Find a solution to the recurrence relation $u_n = u_{n-1} + n$, $n \geq 1$, with $u_0 = 0$.

The solution to $u_n = u_{n-1} + g(n)$ is:

5. Solve the following recurrence relations.
   a) $u_n = u_{n-1} + 2n + 1$, $n \geq 1$, with $u_0 = 7$

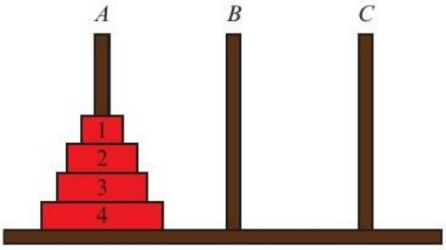
   b) $u_n = u_{n-1} + 5^n$, $n \in \mathbb{N}$, with $u_1 = 3$

Solving $u_n = au_{n-1} + g(n)$: complementary function (C.F.) and particular solution (P.S.):

	Form of particular solution
p with $a \neq 1$	
$pn + q$ with $a \neq 1$	
kp^n with $p \neq a$	
ka^n	
p with $a = 1$	
$pn + q$ with $a = 1$	

6. Solve the recurrence relation $u_n = 3u_{n-1} + 2n$, $n \in \mathbb{Z}^+$, with $u_1 = 3$.

7. The Tower of Hanoi puzzle involves transferring a pile of different sized disks from one peg to another using an intermediate peg. The rules are: only one disk at a time can be moved; a disk can only be moved if it is the top disk on a pile; a larger disk can never be placed on a smaller one.



a) Find the minimum number of moves needed to transfer two disks from one peg to another.

b) Show that three disks can be transferred from one peg to another in 7 moves.

	Disk	From	To
Move 1			
Move 2			
Move 3			
Move 4			
Move 5			
Move 6			
Move 7			

c) Explain why the minimum number of moves, d_n, needed to transfer n disks from one peg to another satisfies the recurrence relation $d_n = 2d_{n-1} + 1$, with $d_1 = 1$.

d) Solve this recurrence relation for d_n.

e) Hence determine the minimum number of moves needed to transfer 15 disks from one peg to another.

4C Solving Second-Order Recurrence Relations

A second-order linear recurrence relation can be written in the form:

1. Consider the recurrence relation $u_n = 2u_{n-1} - u_{n-2}$, $n \geq 2$. Verify that the following particular solutions satisfy this recurrence relation.
   a) $u_n = 3n$

   b) $u_n = 5$

   c) $u_n = 3n + 5$

If $u_n = F(n)$ and $u_n = G(n)$ are both solutions, then:

The auxiliary equation of $u_n = au_{n-1} + bu_{n-2}$, and where it comes from:

	Form of the general solution
Distinct real roots α, β	
Repeated root α	
Complex roots $re^{\pm i\theta}$	

2.
   a) Find a general solution to the recurrence relation $u_n = 2u_{n-1} + 8u_{n-2}$, $n \geq 2$.

   b) Verify that your general solution from part a satisfies the recurrence relation.

c) Given that $u_0 = 4$ and $u_1 = 10$, find a particular solution.

3. Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, with $a_1 = 5$ and $a_2 = 3$.

4. Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$.

5.

a) Find the general solution to the recurrence relation $u_n = 2u_{n-1} - 2u_{n-2}$.

b) Given that $u_0 = 1$ and $u_2 = 2$, find the particular solution to the recurrence relation.

6. The Fibonacci sequence is defined recursively as $F_n = F_{n-1} + F_{n-2}$, $n > 2$, with $F_1 = 1$ and $F_2 = 1$. Find a closed form for F_n.

Solving the non-homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2} + g(n)$:

	Form of particular solution
p with $\alpha, \beta \neq 1$	
pn + q with $\alpha, \beta \neq 1$	
kp^n with $p \neq \alpha, \beta$	
p with $\alpha = 1, \beta \neq 1$	
pn + q with $\alpha = 1, \beta \neq 1$	
p with $\alpha = \beta = 1$	
pn + q with $\alpha = \beta = 1$	
$k\alpha^n$ with $\alpha \neq \beta$	
$k\alpha^n$ with $\alpha = \beta$	

7. Solve the recurrence relation $a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n$, $n \geq 0$, with $a_0 = 2$ and $a_1 = -1$.

8. Find the general solution to $s_n = 3s_{n-1} + 4s_{n-2} + 4^n$.

4D Proving Closed Forms

The steps of a proof by induction for a closed form:

1. The minimum number of moves, d_n, needed to transfer n disks from one peg to another in the Tower of Hanoi problem is given by the recurrence relation $d_n = 2d_{n-1} + 1$, with $d_1 = 1$. Prove, by induction, that $d_n = 2^n - 1$.

2. A sequence u_n satisfies the recurrence relation $u_n = u_{n-1} + n$, with $u_0 = 0$. Prove by induction that $u_n = \frac{n(n+1)}{2}$, $n \geq 0$.

For a second-order recurrence relation, the basis and assumption steps are:

3. A sequence a_n satisfies the recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$, with $a_0 = 4$, $a_1 = 10$. Prove by induction that $a_n = 3(4^n) + (-2)^n$ for all non-negative integers n.

4. The Fibonacci sequence is defined recursively by $u_n = u_{n-1} + u_{n-2}$, $n > 2$, with $u_1 = u_2 = 1$. Show that the closed form for the nth term of the Fibonacci sequence is given by

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

, $n \geq 1$.