

1A The Division Algorithm

If a and b are integers with $a \neq 0$, then b is divisible by a if:

In this case we say that a divides b and write:

If a does not divide b we write:

- Given that $a \mid b$, show that $-a \mid b$.
- For each pair of integers below, determine whether the first integer divides the second.
 - 11, 143
 - 4, 28
 - 15, 47
 - 3, 2
- Find all the divisors of:
 - 8
 - 11

Properties of divisibility. For any $a, b, c \in \mathbb{Z}$, with $a \neq 0$:

- $a \mid a$ and $a \mid 0$ mean:
- $a \mid b$ and $b \mid c \Rightarrow$
- $a \mid b$ and $a \mid c \Rightarrow$
- $a \mid b \Leftrightarrow$
- If a and b are positive integers and $a \mid b$ then

4. Given $a, b, c \in \mathbb{Z}$, prove that if $a \mid b$ and $a \mid c$ then $a \mid bn + cm$ for all $m, n \in \mathbb{Z}$.

Linear combination of b and c :

The division algorithm: if a and b are integers such that $b > 0$, then there exist unique integers q and r such that:

q is called the _____ and r the _____; a is the _____ and b the _____.

Steps to find q and r :

5. Use the division algorithm to find integers q and r such that:

a) $94 = 13q + r$

b) $-232 = 11q + r$

6. Use the division algorithm to prove that, for all integers n , n^2 leaves a remainder of 0 or 1 when divided by 4.

1B The Euclidean Algorithm

If a , b and c are integers and $c \neq 0$, then c is a common divisor of a and b if:

The greatest common divisor, $\gcd(a, b)$, is the positive integer d that satisfies:

1. Find:

- a) $\gcd(3, 12)$ b) $\gcd(25, 25)$ c) $\gcd(90, 84)$

The Euclidean algorithm (given positive integers a and b with $a \geq b$):

2. Use the Euclidean algorithm to find the greatest common divisor of 306 and 657.

Bezout's identity:

3. Use the Euclidean algorithm to find integers x and y such that $306x + 657y = 9$.

4. Use the Euclidean algorithm to find $x, y \in \mathbb{Z}$ such that $\gcd(143, 252) = 143x + 252y$.

Two integers a and b are relatively prime (coprime) if:

The integers a and b are relatively prime if and only if:

5. Find:

a) Use the Euclidean algorithm to show that 49 and 60 are relatively prime.

b) Find integers x and y such that $49x + 60y = 1$

c) Hence find integers p and q such that $49p + 60q = 5$

1C Modular Arithmetic

Let m be a positive integer. If a and b are integers, then a is congruent to b modulo m if:

Notation:

$a \equiv b \pmod{m}$ if and only if a and b leave the same

1. Decide whether the following statements are true or false.
a) $24 \equiv 9 \pmod{5}$ b) $5 \equiv -11 \pmod{8}$ c) $4 \equiv 17 \pmod{2}$

If $a, b \in \mathbb{Z}$, then $a \equiv b \pmod{m}$ if and only if there exists an integer k such that:

2. Given that $n \equiv 10 \pmod{3}$, write down:
a) three different possible values of n

b) the greatest negative value of n

c) the value of n contained in the set $\{0, 1, 2\}$

Properties of congruences:

- $a \equiv 0 \pmod{m}$ if and only if
 - $a \equiv a \pmod{m}$
 - If $a \equiv b \pmod{m}$, then
 - If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then
3. Given that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, prove that $a \equiv c \pmod{m}$.

Rules of arithmetic. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:

- $a \pm c \equiv$
- $ac \equiv$
- $ka \equiv$
- $a^n \equiv$

4. Show that $23^{753} \equiv 1 \pmod{11}$.

5. Find the remainder when 343^5 is divided by 11.

6. Work out the final digit in the number 513^{50} .

7. Find the remainder when 19^{273} is divided by 12.

8. Find the remainder when $1! + 2! + \dots + 100!$ is divided by 15.

1D Divisibility Tests

An integer is divisible by:

- 2 if and only if
- 5 if and only if
- 10 if and only if

A number N with decimal digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$ can be written as:

1. Prove that a positive integer N is divisible by 10 if and only if its last digit is 0.

An integer is divisible by:

- 3 if and only if
- 4 if and only if
- 6 if and only if
- 9 if and only if
- 11 if and only if

2. N is a 3-digit number abc , so that $N = 100a + 10b + c$, where $a, b, c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $a \neq 0$. Prove that $11 \mid N$ if and only if $11 \mid (a - b + c)$.

3. Use the divisibility rules to determine whether:

a) 7146 is divisible by 9

b) 5838 is divisible by 6

c) 2 261 914 is divisible by 4

d) 813 747 is divisible by 11

4. N is a 3-digit number abc , so that $N = 100a + 10b + c$, where a , b and c are integers between 0 and 9, with $a \neq 0$. N has the following properties:

- N is divisible by 9
- The sum of the digits of N is even
- $N \equiv 3 \pmod{11}$

Find all possible values for N , showing your working clearly.

1E Solving Congruence Equations

The set of least residues modulo n is:

Solving a congruence equation means:

1. Solve the equation $x \equiv 86 \pmod{7}$.
2. Solve the equation $x + 15 \equiv 3 \pmod{10}$.

Let $a, b, m \in \mathbb{Z}$, with $m > 0$ and $\gcd(a, m) = d$. For the equation $ax \equiv b \pmod{m}$:

- If $d \nmid b$:

- If $d \mid b$:

3. Explain why the equation $14x \equiv 7 \pmod{22}$ has no solutions.

Cancelling. If $ka \equiv kb \pmod{m}$ and $\gcd(k, m) = 1$, then:

If $ka \equiv kb \pmod{m}$ and $\gcd(k, m) = d$, then:

4. Solve the equation $2n \equiv 26 \pmod{14}$.
5. Solve the equation $42y \equiv 168 \pmod{35}$.

A multiplicative inverse of a modulo m is:

It exists if and only if:

Finding a multiplicative inverse using Bezout's identity:

6.

a) Find a multiplicative inverse of 10 modulo 63.

b) Hence solve the equation $10x \equiv 4 \pmod{63}$.

7. Solve $75x \equiv 12 \pmod{237}$.

Method 1 (back substitution, then multiply by $b \div \gcd(a, m)$):

Method 2 (divide everything, including the modulus, by $\gcd(a, m)$):

1F Fermat's Little Theorem

Fermat's little theorem: if p is a prime number and a is any number not divisible by p , then:

For example: $3^{22} \equiv \underline{\hspace{1cm}} \pmod{23}$ $500^{101} \equiv \underline{\hspace{1cm}} \equiv \underline{\hspace{1cm}} \pmod{101}$

1. Find the least residue of 3^{202} modulo 11.

If a and b are positive integers and p is a prime number with $p \nmid a$, then:

- the multiplicative inverse of a modulo p is
- the solution to the equation $ax \equiv b \pmod{p}$ is

Why:

2. Find the solution to the linear congruence $7x \equiv 10 \pmod{11}$.

3. Find the remainder when 2^{1000} is divided by 13.

4. Find the solution to the congruence equation $x^{103} \equiv 4 \pmod{11}$.

1G Combinatorics

The product rule of counting: if you can choose one item in m different ways, and a second item in n different ways, then:

If you need to choose k items, and the k th item can be chosen in n_k different ways, then:

1. A number plate consists of two letters, followed by two digits, followed by three more letters. The letters can be chosen from the entire alphabet except the letters I, O, Q or Z, and the digits can be chosen from 0 up to 9.



- a) Find the total number of different number plates that can be generated in this way.

All number plates registered in one particular region must start with the letter combinations CW or CX.

- b) Find the total number of different number plates that can be generated in this way.

2. Find the total number of 3-digit even numbers.

3. Find the total number of positive integers less than 1000 that contain the digit 5
 - a) exactly once

- b) at least once

If a set S contains n elements, then the total number of possible subsets of S is:

- 4. A set $S = \{1, 2, 3, 4, 5, 6\}$. Find the total number of possible subsets of S .
- 5. A group of 10 students are being selected to clean up a local park. Given that at least one student must be selected, how many different groups of students could be selected to clean up the park?

The number of different ways of placing n items in order (permutations of n items):

- 6. Consider the letters ABCDEF.
 - a) Find the number of different strings of length 6 that can be written using these letters exactly once each.
 - b) Find the number of these strings that contain:
 - i) the substring DEF
 - ii) the letters DEF together in any order.

The number of possible permutations of r items taken from a set of n items, where $n \geq r$:

7. 15 runners compete in a cross-country race. Find the number of different ways the top 6 positions can be filled.

The number of permutations of n items, of which r are identical:

The number of permutations of n items, of which r_1 are identical, r_2 are identical, and so on:

8. Find the number of possible different permutations of the letters in the word WINNINGS.

The number of possible combinations of r items (in any order) taken from a set of n items, where $n \geq r$:

9. In a lottery game you choose 6 numbers from a choice of 45 by shading in a grid. 6 numbers are selected at random and if you match all six you win a jackpot prize. Find the probability of winning a jackpot prize.



10. In poker, a deck of 52 different cards is used. There are 13 cards in each of four suits. A hand is made up of five cards, arranged in any order.
- a) Work out the total number of different hands that are possible.

A flush is a hand containing five cards of the same suit.

b) Work out the number of hands containing five cards of the same suit.

Four-of-a-kind is a hand containing four cards of the same value, such as four 7s, or four queens.

c) Work out the number of four-of-a-kind hands.

11. The set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Find:

a) the number of subsets of S containing exactly 4 elements

b) the number of different 4-digit numbers that can be made using the elements of S . (Each element can be used only once.)