

2D Isomorphism

Isomorphic groups:

Compare these two groups: the permutations below under composition, and $\{0, 1, 2\}$ under addition modulo 3.

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$\circ$	i	α	β
i	i	α	β
α	α	β	i
β	β	i	α

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

One-to-one function that maps one group onto the other:

Two groups $(G, *)$ and $(H, \circ)$ are isomorphic if there exists a mapping $f: G \rightarrow H$ such that:

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Notation:

- Let $(G, *)$ and $(H, \circ)$ be isomorphic groups with identity elements e_G and e_H respectively, and let $f: G \rightarrow H$ be an isomorphism from G to H . Prove that $f(e_G) = e_H$.

If $f: G \rightarrow H$ is an isomorphism then, for all $a \in G$ and $n \in \mathbb{Z}$:

Group isomorphisms also preserve order and subgroups:

2. The Cayley tables for two isomorphic groups G and H are shown below.

$(G, *)$	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	b	a
d	c	d	a	b

$(H, \circ)$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

a) State the identity element of each group.

b) Describe an isomorphism from G onto H .

3. G and H are cyclic groups with $|G| = |H|$. Prove that $G \cong H$.

Groups of order 8 or less (only these are needed in the exam):

Order	Name	Properties
1	$\mathbb{Z}_1$	
2	$\mathbb{Z}_2$	
3	$\mathbb{Z}_3$	
4	$\mathbb{Z}_4$	
4	Klein four-group, K_4	
5	$\mathbb{Z}_5$	
6	$\mathbb{Z}_6$	
6	S_3, D_6	
7	$\mathbb{Z}_7$	
8	$\mathbb{Z}_8$	
8	D_8	
8	$\mathbb{Z}_4 \times \mathbb{Z}_2$	
8	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	
8	Quaternion group	

Groups of order 8 or less can be classified entirely by:

4. The group G consists of the elements below under the operation of matrix multiplication.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The set $H = \{1, 3, 5, 7, 9, 11, 13, 15\}$ forms a group under multiplication modulo 16.

- a) Show that H contains a subgroup that is isomorphic to G .

- b) Determine whether H is isomorphic to the symmetry group of a square, giving reasons for your answer.

