

2C Order and Subgroups

The order of a finite group G , $|G|$:

The order of an element a , $|a|$:

If $(G, *)$ is finite with $a \in G$, then:

$(G, *)$ is cyclic if and only if:

1. Let $(G, \circ)$ be a finite group. Prove that every element in G must have finite order.
2. For each of the following groups, write down the order of the group, and the order of each element in the group.
 - a) $\{1, -1, i, -i\}$ under complex multiplication
 - b) $\{1, 3, 7, 9\}$ under multiplication modulo 10

3. G has elements $\{e, p, p^2, p^3, q, pq, p^2q, p^3q\}$ under multiplication, where e is the identity. You can assume the associativity axiom is satisfied. A partially completed Cayley table is shown below.

x	e	p	p^2	p^3	q	pq	p^2q	p^3q
e	e	p	p^2	p^3	q	pq	p^2q	p^3q
p	p	p^2	p^3	e	pq	p^2q	p^3q	q
p^2	p^2	p^3	e	p	p^2q	p^3q	q	pq
p^3	p^3	e	p	p^2	p^3q			
q	q	p^3q	p^2q	pq	e			
pq	pq	q	p^3q	p^2q	p			
p^2q	p^2q	pq	q	p^3q	p^2			
p^3q	p^3q	p^2q	pq	q	p^3			

- State the order of the group.
 - State the order of p and q .
 - Complete the Cayley table and verify that G forms a group.
 - Find the order of pq , p^2q and p^3q .
4. The set $\{0, 1, 2, 3, 4\}$ forms a group under addition modulo 5. Explain why no element of this group, other than the identity, can be self-inverse.

Let a be an element in a group $(G, *)$. Then:

- if a has finite order n , then $a^n = e$ if and only if
- if a has infinite order, then
- if $a^x = a^y$ with $x \neq y$, then

Subgroup:

Proper subgroup:

Trivial subgroup:

5.

a) Show that the set $S = \{5^n : n \in \mathbb{Z}\}$ forms a group under multiplication.

b) Determine, with reasons, whether each of the following subsets of S forms a subgroup of $(S, \times)$.

i) $T = \{5^{2n} : n \in \mathbb{Z}\}$

ii) $U = \{5^n : n \in \mathbb{Z}^+\}$

Let G be a group and H a finite non-empty subset of G . Then H is a subgroup of G if:

6. The set $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ forms a group under addition modulo 8. Find two nontrivial proper subgroups of $(S, +_8)$.

$+_8$	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

If G is a finite group, then any element $a \in G$ generates:

Lagrange's theorem:

7. The set $G = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$ forms a group under multiplication modulo 21.

a) Find the elements in the subgroup of $(G, \times_{21})$ generated by the element 5, and state its order.

b) Explain why $(G, \times_{21})$ has no subgroup of order 5.