

2B Cayley Tables and Finite Groups

Finite group:

Cayley table:

Which element comes first in a Cayley table?

1. The set $\{1, -1, i, -i\}$, where $i^2 = -1$, forms a group under multiplication.

a) Draw a Cayley table for this group.

$\times$	1	-1	i	-i
1				
-1				
i				
-i				

b) Write down: i) the identity element ii) the inverse of i iii) the inverse of -1

When a group's elements are displayed in a Cayley table, then:

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-
-
-

Multiplication modulo n , $\times_n$:

Addition modulo n , $+_n$:

The Cayley tables below show the set $\{0, 1, 2, 3, 4\}$ under addition and multiplication modulo 5.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

2. The set $S = \{0, 1, 2, 3, 4\}$. Use the Cayley tables above to determine whether S forms a group under:

a) addition modulo 5

b) multiplication modulo 5

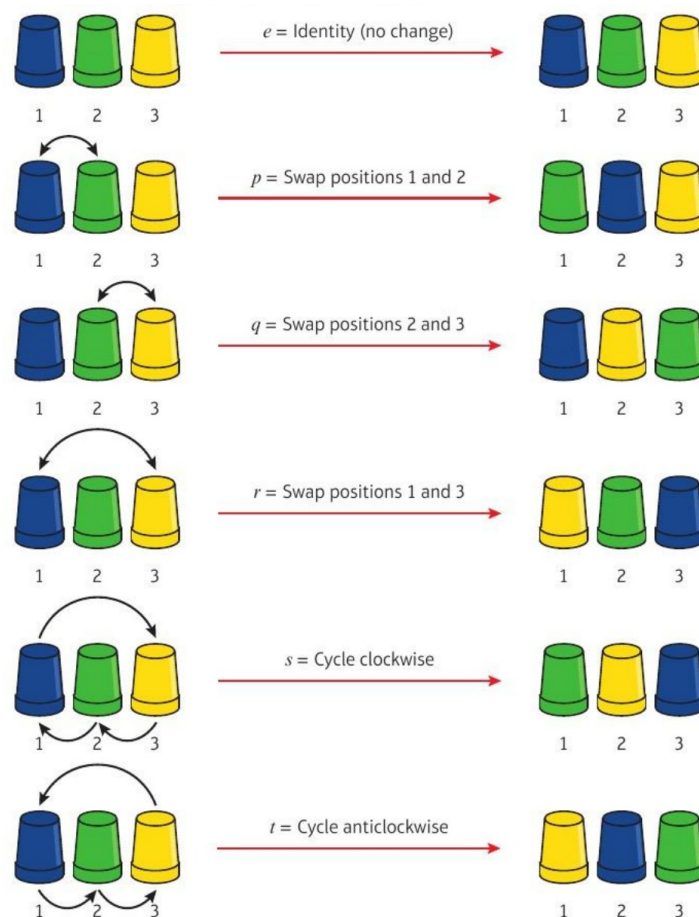
3. $S = \{1, 3, 7, 9\}$. Determine whether each of the following are groups. You may assume that the associative law holds in each case.

a) $(S, \times_{10})$

$\times_{10}$	1	3	7	9
1				
3				
7				
9				

b) $(S, \times_{12})$

Permutation:



Composition: $p \circ q$ means

4. For the set of permutations of 3 cups, $\{e, p, q, r, s, t\}$ as defined above, find:

a) $q \circ p$

b) $r \circ r$

The symmetric group on n elements, S_n :

Two-row notation. The six permutations of 3 cups are:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Composition and inverses in two-row notation. Find $\alpha \circ \beta$, $\beta \circ \alpha$ and α^{-1} for:

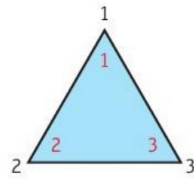
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}$$

5. Show that the permutations below form a group under composition. You may assume the associativity axiom is satisfied.

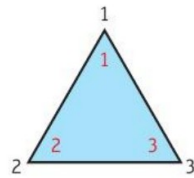
$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

◦	e	p	q	r
e				
p				
q				
r				

Groups of symmetries. The symmetries of an equilateral triangle are:

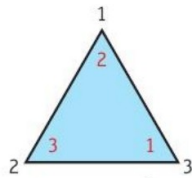


There are three rotational symmetries:



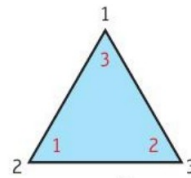
0

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$



Clockwise $\frac{2\pi}{3}$

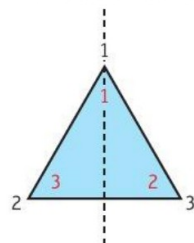
$$R = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$



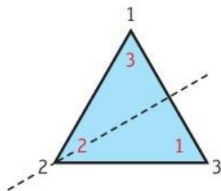
Anticlockwise $\frac{2\pi}{3}$ (or clockwise $\frac{4\pi}{3}$)

$$S = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

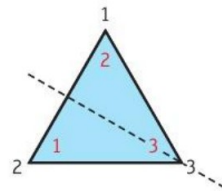
There are three reflections through the three medians:



$$L = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



$$M = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$



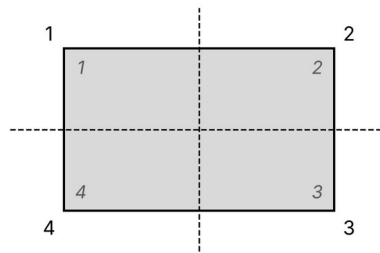
$$N = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

6. Show that $G = \{I, R, S, L, M, N\}$ forms a group under composition of transformations. You may assume that the associative law holds.

$\circ$	I	R	S	L	M	N
I						
R						
S						
L						
M						
N						

Dihedral group D_{2n} :

7. Show that the symmetries of a rectangle form a group under composition of transformations. You may assume that the associativity axiom holds.



Cyclic group:

Generator:

8. The set $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is a group under addition modulo 8. Show that 3 is a generator of this group and write each element in terms of this generator.
9. The set $\{1, 3, 5, 7\}$ forms a group under multiplication modulo 8. Show that this group is not cyclic.

