

7A Forming Recurrence Relations

A recurrence relation:

An initial condition:

First-order and second-order recurrence relations:

1. Harry owes £500 on a credit card that charges 1.5% interest each month. He decides to make no new charges and pays off £50 each month. Formulate a recurrence relation that describes the balance remaining on the credit card after n months.
2. The deer population of a county was observed to be 1200 in a given year. The population is modelled to increase at a rate of 15% each year. Let d_n be the population of deer n years later. Explain why the deer population is modelled by the recurrence relation $d_n = 1.15d_{n-1}$, with $d_0 = 1200$.
3. (A level only) A population of bacteria has initial size 200. After one hour, the population has reached 220. The population grows in such a way that the rate of growth doubles each hour. Write a recurrence relation to describe the number of bacteria, b_n , after n hours.

How to verify that a general term $u_n = f(n)$ satisfies a recurrence relation:

4. A sequence has the general term $u_n = 3n - 1$. Verify that the sequence satisfies the recurrence relation $u_n = 3 + u_{n-1}$.

5. A sequence has the general term $u_n = 2 \times 3^{n-1}$. Verify that the sequence satisfies the recurrence relation $u_n = 3u_{n-1}$.

Recursive form and closed form of a sequence:

7B Solving First-Order Recurrence Relations

The order of a recurrence relation:

A first-order linear recurrence relation can be written in the form:

It is homogeneous when:

Back substitution:

1. Find a closed form for the sequence $a_n = 5a_{n-1}$, $n > 0$; $a_0 = 1$.

The solution to the first-order homogeneous recurrence relation $u_n = au_{n-1}$ is:

2. Solve the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$, with $a_0 = 3$.

General solution and particular solution:

3. Solve the recurrence relation $a_n = -3a_{n-1}$, $n \geq 1$, with $a_1 = 6$.

4. Find a solution to the recurrence relation $u_n = u_{n-1} + n$, $n \geq 1$, with $u_0 = 0$.

The solution to $u_n = u_{n-1} + g(n)$ is:

5. Solve the following recurrence relations.

a) $u_n = u_{n-1} + 2n + 1, n \geq 1$, with $u_0 = 7$

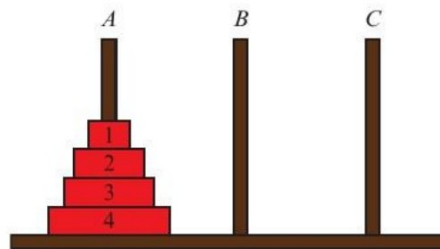
b) $u_n = u_{n-1} + 5^n, n \in \mathbb{N}$, with $u_1 = 3$

Solving $u_n = au_{n-1} + g(n)$: complementary function (C.F.) and particular solution (P.S.):

	Form of particular solution
p with $a \neq 1$	
$pn + q$ with $a \neq 1$	
kp^n with $p \neq a$	
ka^n	
p with $a = 1$	
$pn + q$ with $a = 1$	

6. Solve the recurrence relation $u_n = 3u_{n-1} + 2n, n \in \mathbb{Z}^+$, with $u_1 = 3$.

7. The Tower of Hanoi puzzle involves transferring a pile of different sized disks from one peg to another using an intermediate peg. The rules are: only one disk at a time can be moved; a disk can only be moved if it is the top disk on a pile; a larger disk can never be placed on a smaller one.



- a) Find the minimum number of moves needed to transfer two disks from one peg to another.
- b) Show that three disks can be transferred from one peg to another in 7 moves.

	Disk	From	To
Move 1			
Move 2			
Move 3			
Move 4			
Move 5			
Move 6			
Move 7			

- c) Explain why the minimum number of moves, d_n , needed to transfer n disks from one peg to another satisfies the recurrence relation $d_n = 2d_{n-1} + 1$, with $d_1 = 1$.

- d) Solve this recurrence relation for d_n .

- e) Hence determine the minimum number of moves needed to transfer 15 disks from one peg to another.

7C Solving Second-Order Recurrence Relations

A second-order linear recurrence relation can be written in the form:

1. Consider the recurrence relation $u_n = 2u_{n-1} - u_{n-2}$, $n \geq 2$. Verify that the following particular solutions satisfy this recurrence relation.

a) $u_n = 3n$

b) $u_n = 5$

c) $u_n = 3n + 5$

If $u_n = F(n)$ and $u_n = G(n)$ are both solutions, then:

The auxiliary equation of $u_n = au_{n-1} + bu_{n-2}$, and where it comes from:

	Form of the general solution
Distinct real roots α, β	
Repeated root α	
Complex roots $re^{\pm i\theta}$	

2.

- a) Find a general solution to the recurrence relation $u_n = 2u_{n-1} + 8u_{n-2}$, $n \geq 2$.

- b) Verify that your general solution from part a satisfies the recurrence relation.

c) Given that $u_0 = 4$ and $u_1 = 10$, find a particular solution.

3. Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, with $a_1 = 5$ and $a_2 = 3$.

4. Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$.

5.

a) Find the general solution to the recurrence relation $u_n = 2u_{n-1} - 2u_{n-2}$.

b) Given that $u_0 = 1$ and $u_2 = 2$, find the particular solution to the recurrence relation.

6. The Fibonacci sequence is defined recursively as $F_n = F_{n-1} + F_{n-2}$, $n > 2$, with $F_1 = 1$ and $F_2 = 1$. Find a closed form for F_n .

Solving the non-homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2} + g(n)$:

	Form of particular solution
p with $\alpha, \beta \neq 1$	
$pn + q$ with $\alpha, \beta \neq 1$	
kp^n with $p \neq \alpha, \beta$	
p with $\alpha = 1, \beta \neq 1$	
$pn + q$ with $\alpha = 1, \beta \neq 1$	
p with $\alpha = \beta = 1$	
$pn + q$ with $\alpha = \beta = 1$	
$k\alpha^n$ with $\alpha \neq \beta$	
$k\alpha^n$ with $\alpha = \beta$	

7. Solve the recurrence relation $a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n$, $n \geq 0$, with $a_0 = 2$ and $a_1 = -1$.

8. Find the general solution to $s_n = 3s_{n-1} + 4s_{n-2} + 4^n$.