

7A And You Thought Chapter 6 Was Complicated...

1. A grower specialising in cut flowers is considering cultivating four varieties of new 'Sunlip' flowers A, B, C and D in one of his fields. He estimates the time, in hours per hectare, taken for each of four stages – sowing, thinning, picking and packing for each variety.

For variety A sowing, thinning, picking and packing will take 3, 18, 20 and 24 hours per hectare respectively.

For variety B sowing, thinning, picking and packing will take 4, 17, 25 and 27 hours per hectare respectively.

For variety C sowing, thinning, picking and packing will take 3, 19, 26 and 28 hours per hectare respectively.

For variety D sowing, thinning, picking and packing will take 5, 16, 22 and 23 hours per hectare respectively.

For this crop he can devote up to 70 hours to sowing, 360 to thinning, 500 to picking and 550 to packing.

He estimates the total profit, in pounds per hectare, as 67, 63, 71 and 75 for varieties A, B, C and D respectively.

He has up to 20 hectares to use for this crop and wishes to maximise his profit.

Formulate this as a linear programming problem. Define your variables, state your objective and write your constraints as inequalities.

2. In order to supplement his diet Andy wishes to take some Vitatab, Weldo, Xtramin and Yestivit tablets. Amongst other ingredients, the contents of vitamins A, B, C and iron, in milligrams per tablet, are shown in the table.

	A	B	C	Iron
Vitatab	10	10	20	4
Weldo	15	20	10	5
Xtramin	25	15	15	3
Yestivit	20	15	20	2

Andy wishes to take tablets to provide him with at least 80, 30, 60 and 14 milligrams of vitamins A, B, C and iron per day.

Because of other factors Andy wants at least 25% of the tablets he takes to be Vitatab and wants to take at least twice as many Weldo as Yestivit.

The costs of the tablets are 4, 6, 12 and 7 pence per tablet. Andy wishes to minimise the cost.

Formulate this as a linear programming problem, defining your variables, stating your objective and writing your constraints as inequalities.

Slack variables:

3. Rewrite the inequality

As an equation, using the slack variable, .

4. Rewrite the constraints for Example 1 as equations using slack variables and

7B The Simplex Method

In chapter 6 we used graphs to help visualise where each vertex would be with two variables. This is much harder with three variables and dang near impossible with four or more. The simplex method:

- Starts at one vertex
- Moves between vertices in sequence, increasing the objective function and one variable at a time, until it is optimal
- Slack variables are key

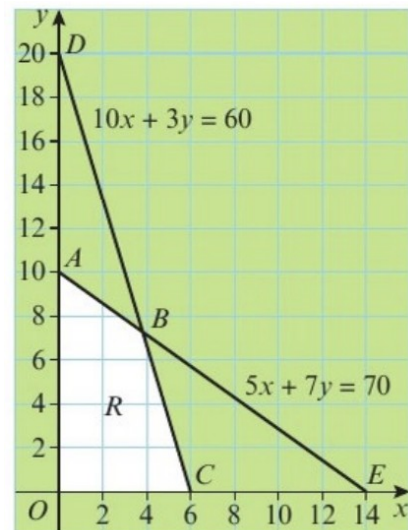
1.

- a) Explain the significance of the slack variables in the graphical representation of the linear programming problem below.

Maximise

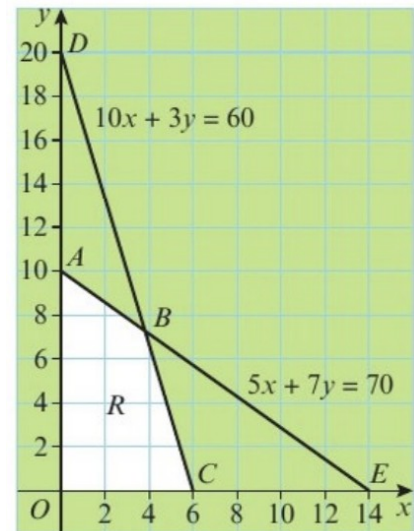
Subject to:

The feasible region, R, is shown.



At each vertex, two of the four vertices are zero.

- b) Explain, in detail, how the algebraic simplex algorithm is used to solve the above problem, relating each stage to the given graph.



2. Show, in detail, the application of the algebraic simplex algorithm to the linear programming problem below.

Maximise

Subject to :

3. Solve the linear programming problem in Example 1 using simplex tableaux (plural of tableau).

Maximise

Subject to:

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Always start at $x=0$, $y=0$ and follow the steps:

- 1** Draw the tableaux.
You need a basic variable column on the left, one column for each variable (including the slack variables) and a value column. You need one row for each constraint and the bottom row for the objective function.
- 2** Create the initial tableau.
Enter the coefficients of the variables in the appropriate column and row.
- 3** Look along the objective row for the most negative entry: this indicates the pivot column.
- 4** Calculate the θ values, for each of the constraint rows, where
 $\theta = (\text{the term in the value column}) \div (\text{the term in the pivot column})$
- 5** Select the row with the smallest, positive θ value to become the pivot row.
- 6** The element in the pivot row and pivot column is the pivot.
- 7** Divide the row found in step **5** by the pivot, and change the basic variable at the start of the row to the variable at the top of the pivot column. This is now the pivot row.
- 8** Use the pivot row to eliminate the pivot's variable from the other rows.
This means that the pivot column now contains one 1 and zeros.
- 9** Repeat steps **3** to **8** until there are no more negative numbers in the objective row.
- 10** The tableau is now optimal and the non-zero values can be read off using the basic variable column and value column.

4. Minimise

Subject to:

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

Basic variable	x	y	r	s	Value

- 5.
- a) Use simplex tableaux to solve the linear programming problem below (from Example 2).

Maximise

Subject to:

Basic variable						Value

Basic variable						Value

Basic variable						Value

Basic variable						Value

b) Verify your solution using the original problem.

6.

a) Use the simplex tableau method to solve the following linear programming problem.

Maximise

Subject to:

Basic variable							Value	

Basic variable							Value	

Basic variable							Value	

Basic variable							Value	

Basic variable							Value	

Basic variable							Value	

b) State the values of the objective function and every variable at the optimal point.

c) Write down the equations given by your optimal tableau.

d) Use the profit equation you wrote down in part c to explain why your final tableau is optimal.

7C Integer Solutions

Consider integer solutions near the optimal solution

- Recall (1/3) from 7B solutions:

Basic variable	x	y	r	s	Value
y	0	1	$\frac{2}{11}$	$-\frac{1}{11}$	$\frac{80}{11}$
x	1	0	$-\frac{3}{55}$	$\frac{7}{55}$	$\frac{42}{11}$
P	0	0	$\frac{1}{5}$	$\frac{1}{5}$	26

$\times$ $\times$
 $\bullet$
 $\times$ $\times$

Point	$5x + 7y \leq 70$	$10x + 3y \leq 60$	In feasible region?	$P = 3x + 2y$
(3, 7)				
(3, 8)				
(4, 7)				
(4, 8)				

2. Solve the linear programming model from example 2 in 7B

Maximise

Subject to :

Basic variable	x	y	z	r	s	Value
y	1	1	0	$\frac{1}{2}$	0	$2\frac{1}{2}$
z	$\frac{1}{2}$	0	1	$-\frac{3}{8}$	$\frac{1}{4}$	$1\frac{7}{8}$
P	6	0	0	3	2	45

Point	$2x + 2y \leq 5$	$5x + 3y + 4z \leq 15$	In feasible region?	$P = 10x + 12y + 8z$
(0, 2, 1)				
(0, 2, 2)				
(0, 3, 1)				
(0, 3, 2)				

7D Two-Stage Simplex Method

What if we have constraints?

- (0,0) is no longer in the feasible region

Slack variables can't "just" be taken away instead. Eg.

Would become

(s is now called a surplus value)

But this means when

Which contradicts

We therefore introduce ANOTHER "artificial" variable such as

Which can now be written in a simplex tableau

1. Use surplus and artificial variables to write the following inequalities as equations.

- 1 Use slack, surplus and artificial variables, as necessary, to write all the constraints as equations.
- 2 Define a new objective function to minimise the sum of all the artificial variables.
- 3 Use the simplex method to solve this problem.
- 4 If the minimum sum of the artificial values is 0 then the solution found is a basic feasible solution of the original problem, which is then the starting point for the second stage. Use the simplex method again to solve this problem.
- 5 If the minimum sum of the artificial variables is not 0 then the original problem has no feasible solution.

Notation This method is called the **two-stage simplex method** because you have two separate applications of the simplex algorithm. Steps **1** to **3** given here constitute the first stage.

Step 2 is the **really** new bit! MINIMISE sum of artificial variables so

If this is zero, then a solution exists (so simplex to this row in tableau)

2. Minimise

Subject to

Basic variable	x	y	z	s_1	s_2	s_3	Value

Basic variable	x	y	z	s_1	s_2	s_3	Value

Basic variable	x	y	z	s_1	s_2	s_3	Value

3. Verify that the following linear programming problem has no feasible solution.

Maximise

Subject to

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	

7E The Big-M Method

is an arbitrarily large number so things like:

is positive

is negative

1. Solve the following linear programming problem using the Big-M method.

Maximise

Subject to:

Basic variable								

Basic variable								

Basic variable								

Basic variable								

Basic variable								

- 1 Introduce a slack variable for each constraint of the form $\leq$.
- 2 Introduce a surplus variable and an artificial variable for each constraint of the form $\geq$.
- 3 For each artificial variable $a_1, a_2, a_3 \dots$ subtract $M(a_1 + a_2 + a_3 \dots)$ from the objective function, where M is an arbitrarily large number.
- 4 Eliminate the artificial variables from your objective function so that the variables remaining in your objective function are non-basic variables.
- 5 Formulate an initial tableau, and apply the simplex method in the normal way.

Hint In the Big-M method, any artificial variables will always start as basic variables.

Step 3 is the new bit here.

2. Use the Big-M method to solve this linear programming problem.

Minimise

Subject to:

Basic variable								

Basic variable								

Basic variable								

Basic variable								

3. Set up the initial tableau for solving the following linear programming problem using the Big-M method:

Maximise

Subject to

2

(Draw your own tables!)