QQQ – Statistics Yr2 - Chapter 1 – More complex numbers

Total Marks: 24

(24 = Platinum, 22 = Gold, 19 = Silver, 17 = Bronze)

1. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

2. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\text{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \text{i}y$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

3. The infinite series C and S are defined by

$$\text{C} = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$\text{S} = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$\text{C} + \text{iS} = \frac{2\text{e}^{\text{i}\theta}}{2 - \text{e}^{4\text{i}\theta}}$$

(4)

(b) Hence show that

$$\text{S} = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

Solutions

(a)	$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1	2.1
	$= 2\cos n\theta$*	A1*	1.1b
		(2)	
b)	$\left(z + z^{-1}\right)^4 = 16\cos^4\theta$	B1	2.1
	$\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1	2.1
	$= z^4 + z^{-4} + 4\left(z^2 + z^{-2}\right) + 6$	A1	1.1b
	$= 2\cos 4\theta + 4\left(2\cos 2\theta\right) + 6$	M1	2.1
	$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$*	A1*	1.1b
		(5)	

(7 marks)

2.

(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2i)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+i\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+i\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}e^{i\arctan\left(\frac{2}{6}\right)}e^{i\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)i$	A1	1.1b
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)i$	A1	1.1b
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2i)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+i\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+i\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}e^{i\arctan\left(\frac{2}{6}\right)}e^{i\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)i$	A1	1.1b
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)i$	A1	1.1b
		(6)	
(b) Way 1	Area $ABC = 3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area $AOB = \frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1	2.1
	Area $DEF = \frac{1}{4}ABC$ or $\frac{3}{4}AOB$	**d**M1	3.1a
	$=\frac{3}{8}\times 40\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	

Many other ways possible. See full ms from 2019 for additional solutions

3.

a) ay 1	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos 5\theta+\mathrm{i}\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+\mathrm{i}\sin 9\theta)+\ldots\right)$	M1	1.1b
	$=\mathrm{e}^{\mathrm{i}\theta}+\frac{1}{2}\mathrm{e}^{5\mathrm{i}\theta}\left(+\frac{1}{4}\mathrm{e}^{9\mathrm{i}\theta}+\ldots\right)$	A1	2.1
	$\mathrm{C}+\mathrm{iS}=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{4\mathrm{i}\theta}}$	M1	3.1a
	$=\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}$ *	A1*	1.1b
		(4)	
a) ay 2	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos 5\theta+\mathrm{i}\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+\mathrm{i}\sin 9\theta)+\ldots\right)$	M1	1.1b
	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)^5\left(+\frac{1}{4}(\cos\theta+\mathrm{i}\sin\theta)^9+\ldots\right)$	A1	2.1
	$\mathrm{C}+\mathrm{iS}=\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)^4}=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{4\mathrm{i}\theta}}$	M1	3.1a
	$=\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}$ *	A1*	1.1b
		(4)	

(b) Way 1	$\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}\times\frac{2-\mathrm{e}^{-4\mathrm{i}\theta}}{2-\mathrm{e}^{-4\mathrm{i}\theta}}$	M1	3.1a
	$\frac{4\mathrm{e}^{\mathrm{i}\theta}-2\mathrm{e}^{-3\mathrm{i}\theta}}{4-2\mathrm{e}^{-4\mathrm{i}\theta}-2\mathrm{e}^{4\mathrm{i}\theta}+1}$	A1	1.1b
	$\frac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos 3\theta+2\mathrm{i}\sin 3\theta}{5-2\cos 4\theta+2\mathrm{i}\sin 4\theta-2\cos 4\theta-2\mathrm{i}\sin 4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\frac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b
		(4)	
(b) Way 2	$\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}=\frac{2(\cos\theta+\mathrm{i}\sin\theta)}{2-(\cos 4\theta+\mathrm{i}\sin 4\theta)}\times\frac{2-(\cos 4\theta-\mathrm{i}\sin 4\theta)}{2-(\cos 4\theta-\mathrm{i}\sin 4\theta)}$	M1	3.1a
	$\frac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos\theta\cos 4\theta-2\sin\theta\sin 4\theta+2\mathrm{i}\sin 4\theta\cos\theta-2\mathrm{i}\sin\theta\cos 4\theta}{4+\cos^2 4\theta+\sin^2 4\theta-4\cos 4\theta}$	A1	1.1b
	$\frac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos 3\theta+2\mathrm{i}\sin 3\theta}{5-2\cos 4\theta+2\mathrm{i}\sin 4\theta-2\cos 4\theta-2\mathrm{i}\sin 4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\frac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b