QQQ – Core Pure 2 - Chapter 5 – Polar Coordinates

Total Marks: 21

(21 = Platinum, 19 = Gold, 17 = Silver, 15 = Bronze)

1.

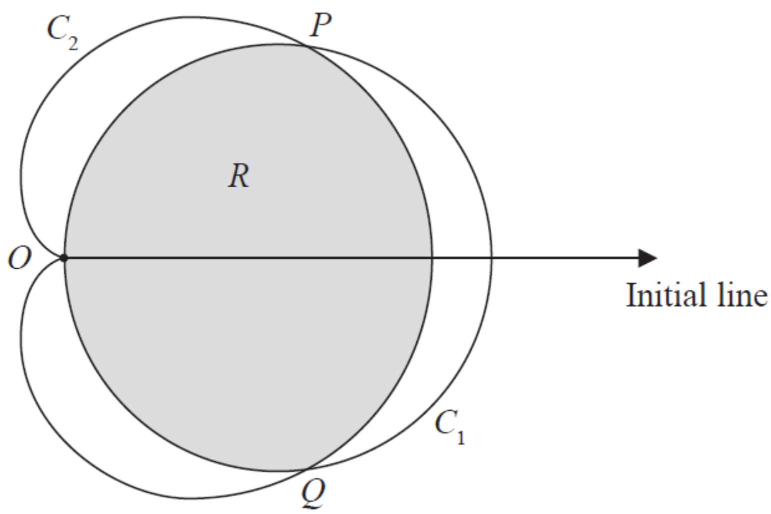

Figure 1

The curve C_1 with equation

$$r = 7\cos\theta, \quad -\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$$

and the curve C_2 with equation

$$r = 3(1 + \cos\theta), \quad -\pi < \theta \leqslant \pi$$

are shown on Figure 1.

The curves C_1 and C_2 both pass through the pole and intersect at the point P and the point Q.

(a) Find the polar coordinates of P and the polar coordinates of Q.

(3)

The regions enclosed by the curve C_1 and the curve C_2 overlap, and the common region R is shaded in Figure 1.

(b) Find the area of R.

(7)

2.

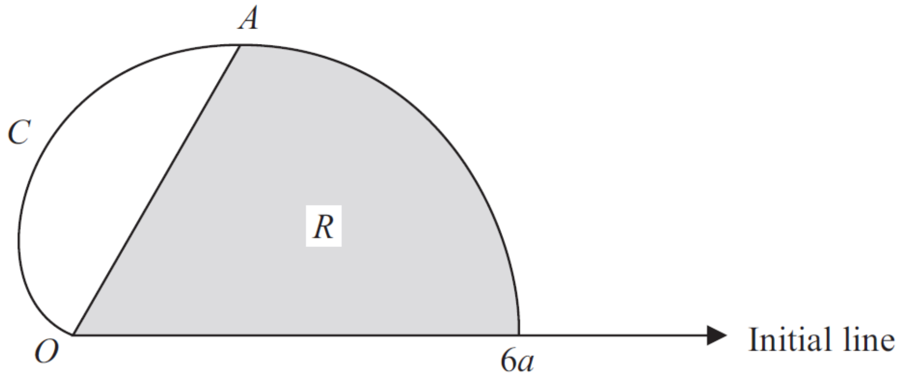

Figure 1

The curve C, shown in Figure 1, has polar equation

$$r = 3a\,(1 + \cos\theta), \qquad 0 \leqslant \theta < \pi$$

The tangent to C at the point A is parallel to the initial line.

(a) Find the polar coordinates of A.

(6)

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line OA.

(b) Use calculus to find the area of the shaded region R, giving your answer in the form $a^2\left(p\pi + q\sqrt{3}\right)$, where p and q are rational numbers.

(5)

Solutions

8(a)	$7\cos\theta = 3+3\cos\theta \Rightarrow \cos\theta = \frac{3}{4} \quad \theta = ...$	Solve to $\theta = ...$	M1
	$P(\frac{21}{4},\alpha)$ and $Q(\frac{21}{4},-\alpha)$ or $\left(\frac{21}{4},2\pi-\alpha\right)$ where $\alpha = \arccos\frac{3}{4}$ or $0.7227...$	A1: Angles correct, Decimal for α to be 3 sf minimum A1: r to be $\frac{21}{4}, 5\frac{1}{4}$ or 5.25 Need not be in coordinate brackets	A1 A1cao (3)

(b)	$\left(2\times\frac{1}{2}\right)\int (7\cos\theta)^2 \mathrm{d}\theta$ $=\frac{49}{2}\int(\cos 2\theta+1)\mathrm{d}\theta$	M1: Use of $\frac{1}{2}\int r^2 \mathrm{d}\theta$ for C_1 leading to $k\int(\cos 2\theta \pm 1)\mathrm{d}\theta$ OR Area of a segment	M1
	$=\frac{49}{2}\left[\frac{1}{2}\sin 2\theta+\theta\right]$ OR Area of sector $=\frac{1}{2}r^2(\beta-\sin\beta)$ $=\frac{1}{2}\left(\frac{7}{2}\right)^2(\pi-2\alpha-\sin(\pi-2\alpha))$	A1: Correct integration of $49\cos^2\theta$ Ignore any limits shown. OR correct expression for the area of the sector	A1
	$\left(2\times\frac{1}{2}\right)\int (3+3\cos\theta)^2 \mathrm{d}\theta$ $=9\int\left(1+2\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$ $=9\int\left(1+2\cos\theta+\frac{1}{2}(\cos 2\theta+1)\right)\mathrm{d}\theta$	M1: Set up the integral for C_2 and reach $k\int\left(1+2\cos\theta+\frac{1}{2}(\cos 2\theta \pm 1)\right)\mathrm{d}\theta$	M1
	$=\left[9\theta+18\sin\theta+\frac{9}{2}(\frac{1}{2}\sin 2\theta+\theta)\right]_0^{\alpha}$	dM1: Correct integration of the trig functions $\cos\theta \to \pm\sin\theta$, $\cos 2\theta \to \pm\frac{1}{2}\sin 2\theta$ A1 Fully correct integration with limits, $0 \to \alpha$ or $-\alpha \to \alpha$	dM1 A1
	Area $=9\alpha+18\sin\alpha+\frac{9}{2}(\frac{1}{2}\sin 2\alpha+\alpha)$ $+\frac{49\pi}{4}-\frac{49}{2}(\frac{1}{2}\sin 2\alpha+\alpha)$	dM1: **Depends on all 3 M marks above** Combine areas correctly to find the required area. Use of correct limits required.	dM1
	$\frac{49\pi}{4}+\frac{3\sqrt{7}}{4}-11\alpha$ or 32.5	A1cso Correct answer, exact or awrt 32.5	A1cso (7) **Total 10**

2.

6 (a)	$r\sin\theta = 3a\sin\theta + 3a\sin\theta\cos\theta$ OR $3a\sin\theta + \frac{3}{2}a\sin 2\theta$	M1
	$\frac{\mathrm{d}(r\sin\theta)}{\mathrm{d}\theta} = 3a\cos\theta + 3a\cos^2\theta - 3a\sin^2\theta$ $3a\cos\theta + 3a\cos 2\theta$	dM1
	$2\cos^2\theta + \cos\theta - 1 = 0$ terms in any order	A1
	$(2\cos\theta - 1)(\cos\theta + 1) = 0$	
	$\cos\theta = \frac{1}{2}$ $\theta = \frac{\pi}{3}$ $(\theta = \pi$ need not be seen$)$	ddM1A1
	$r = 3a \times \frac{3}{2} = \frac{9}{2}a$	A1 (6)

(a)M1 using $r\sin\theta$ $r\cos\theta$ scores M0

dM1 Attempt the differentiation of $r\sin\theta$, inc use of product rule or $\sin 2\theta = 2\sin\theta\cos\theta$

A1 Correct 3 term quadratic in $\cos\theta$

ddM1 dep on both M marks. Solve their quadratic (usual rules) giving one or two roots

A1 Correct quadratic solved to give $\theta = \frac{\pi}{3}$

A1 Correct r obtained No need to see coordinates together in brackets
Special Case: If $r\cos\theta$ used, score M0M1A0M0A0A0to

(b)	Area $= \frac{1}{2}\int r^2 \mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} 9a^2(1+\cos\theta)^2 \mathrm{d}\theta$	
	$= \frac{9a^2}{2}\int_0^{\frac{\pi}{3}}(1 + 2\cos\theta + \cos^2\theta)\mathrm{d}\theta$	M1
	$= \frac{9a^2}{2}\int_0^{\frac{\pi}{3}}\left(1 + 2\cos\theta + \frac{1}{2}(\cos 2\theta + 1)\right)\mathrm{d}\theta$	M1
	$= \frac{9a^2}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right)\right]_0^{\frac{\pi}{3}}$	dM1A1
	$\frac{9a^2}{2}\left[\frac{\pi}{3} + \sqrt{3} + \frac{1}{4} \times \frac{\sqrt{3}}{2} + \frac{\pi}{6} \quad (-0)\right]$	
	$\frac{9a^2}{2}\left[\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right] = \left(\frac{9\pi}{4} + \frac{81\sqrt{3}}{16}\right)a^2$	A1 (5) [11]

(b)M1 Use of correct area formula, $\frac{1}{2}$ may be seen later, inc squaring the bracket to obtain 3 terms - limits need not be shown.

M1 Use double angle formula (formula to be of form $\cos^2\theta = \pm\frac{1}{2}(\cos 2\theta \pm 1)$) to obtain an integrable function - limits need not be shown, $\frac{1}{2}$ from area formula may be missing,

dM1 attempt the integration - limits not needed – dep on 2nd M mark but not the first

A1 correct integration – substitution of limits not required

A1 correct final answer any equivalent provided in the demanded form.