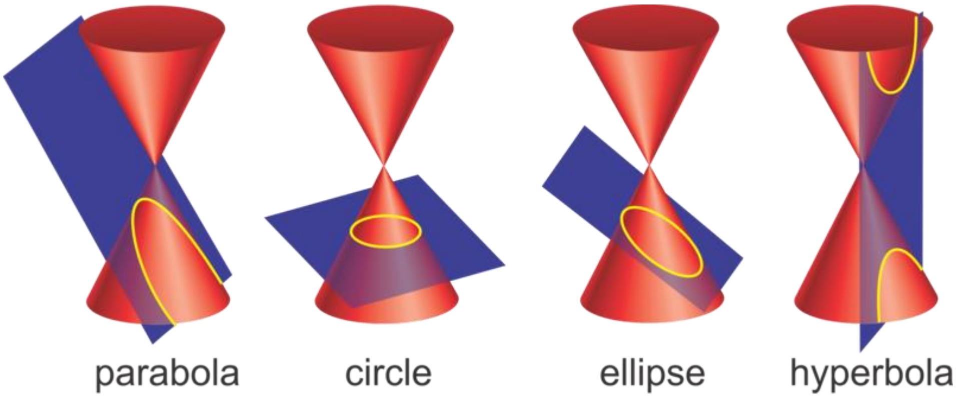
3A Ellipses



1. The ellipse E has equation
   $4x^2 + 9y^2 = 36.$

a) Sketch E.

b) Write down its parametric equations.

2. The ellipse E has parameter equations:

$$x = 3\cos\theta \ y = 5\sin\theta$$

Determine its Cartesian equation.

3B Hyperbolas

1. The hyperbola H has equation
   $4x^2 - 9y^2 = 36.$

a) Sketch H.

b) Write down the equations of the asymptotes of H.

c) Find parametric Equations for H.

2. A hyperbola H has parametric equations

$$x = 4\sec t, \qquad y = \tan t, \qquad -\pi \leq t < \pi, \qquad t \neq \pm\frac{\pi}{2}$$

a) Find a Cartesian equation for H.

b) Sketch H.

c) Write down the equations of the asymptotes of H.

3C Eccentricity

Ellipse

Parabola

Hyperbola

1. Show that for $0 < e < 1$, the ellipse with focus $\mathrm{S} = (ae, 0)$ and directrix $x = \frac{a}{e}$ has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2.

a) Find foci of the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and give the equation of the directrices. Hence sketch the ellipse.

b) Find foci of the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and give the equation of the directrices. Hence sketch the ellipse.

3. If P is a point on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $PS + PS' = 2a$

4. Show that for $e > 1$, the hyperbola with foci at $(\pm ae, 0)$ and directrices at $x = \pm\frac{a}{e}$ has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

5.

a) Sketch the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$, indicating the foci, directrices and equations of the asymptotes.

b) Sketch the hyperbola with equation $\frac{x^2}{16} - \frac{y^2}{25} = 1$, indicating the foci, directrices and equations of the asymptotes.

A quick note on hyperbolas and ellipses:

For hyperbolas, you don't care which of a and b are bigger. For ellipses, swapping the a and b has the effect of rotating the ellipse 90° and hence the foci/directrices too. We don't get this same rotation for hyperbolas.

Formula book p19

Conics

	Ellipse	**Parabola**	**Hyperbola**	**Rectangular Hyperbola**
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e<1$ $b^2=a^2(1-e^2)$	$e=1$	$e>1$ $b^2=a^2(e^2-1)$	$e=\sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x=\pm\frac{a}{e}$	$x=-a$	$x=\pm\frac{a}{e}$	$x+y=\pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a}=\pm\frac{y}{b}$	$x=0, y=0$

Note:

Tangent:	$ty=x+at^2$	$x+t^2y=2ct$
Normal:	$y+tx=2at+at^3$	$t^3x-ty=c(t^4-1)$

3D Gradients of an Ellipse

1. Find the equation of the tangent to the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at the point $(3\cos\theta, 2\sin\theta)$

2. Show that the equation of the normal to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ is $by\cos\theta = ax\sin\theta + (b^2 - a^2)\cos\theta\sin\theta$

Notes on tangents and normals to an ellipse

3. The point $P\left(2, \frac{3\sqrt{3}}{2}\right)$ lies on the ellipse E with parametric equations $x = 4\cos\theta\,,\ y = 3\sin\theta$.

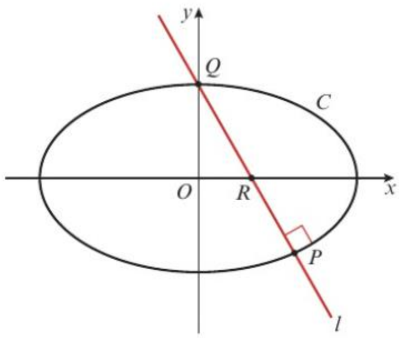
a) Find the value of θ at the point P.

b) Find the equation of the normal to the ellipse at point P.

4. Show that the condition for $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$a^2m^2 + b^2 = c^2$$

5. The ellipse C has the equation $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$. The line l is normal to the ellipse at the point P and passes through the point Q, where C cuts the y-axis, as shown in the diagram.
Find the exact coordinates of the point R, where l cuts the positive x-axis.



3E Gradients of a Hyperbola

1. Find the equation of the tangent to the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$ at the point $(6, 2\sqrt{3})$

2. Prove that the equation of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \cosh t\,,\ b \sinh t)$ is $\boldsymbol{ay \sinh t + ab = bx \cosh t}$

3. Show that an equation of the normal to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$ is $by + ax\sin\theta = (a^2 + b^2)\tan\theta$.

4. Show that the condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that m and c satisfy $b^2 + c^2 = a^2m^2$.

5. The tangent to the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$ at the point $(3\cosh t\,,\ 2\sinh t)$ crosses the y-axis at the point $(0, -1)$. Find the value of t.

6. The hyperbola H has equation $\frac{x^2}{36} - \frac{y^2}{9} = 1$.
The line l_1 is the tangent to H at the point $P(6\cosh t\,, 3\sinh t)$. The line l_2 passes through the origin and is perpendicular to l_1. The lines l_1 and l_2 intersect at the point Q.
Show that the coordinates of the point Q are $\left(\frac{6\cosh t}{4\sinh^2 t + \cosh^2 t}, -\frac{12\sinh t}{4\sinh^2 t + \cosh^2 t}\right)$.

3F More Loci

1. The tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos t, b\sin t)$ crosses the x-axis at A and the y-axis at B.
   Find an equation for the locus of the mid-point of AB as P moves around the ellipse.

2. The normal at $P(ap^2,\ 2ap)$ and the normal at $Q(aq^2,\ 2aq)$ to the parabola with equation $y^2 = 4ax$ meet at R.

a) Find the coordinates of R.

The chord PQ passes through the focus $(a, 0)$ of the parabola.

b) Show that $pq = -1$

c) Show that the locus of R is a parabola with equation $y^2 = a(x - 3a)$