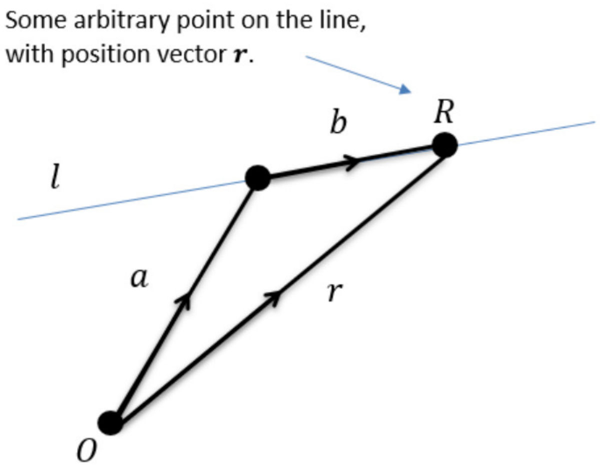
Vector equations of lines



✎ Equation of straight line:

$$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = 0$$
$$or\ \boldsymbol{r} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$$

Find the vector equation of the line through the points $(1,2,-1)$ and $(3,-2,2)$ in the form $(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = 0$ and in the form $\boldsymbol{r} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$.

4. The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

The point A has coordinates $(4, -3, c)$

(a) Find, in terms of c, an equation for the line that passes through the point A and is perpendicular to the plane Π. Give your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

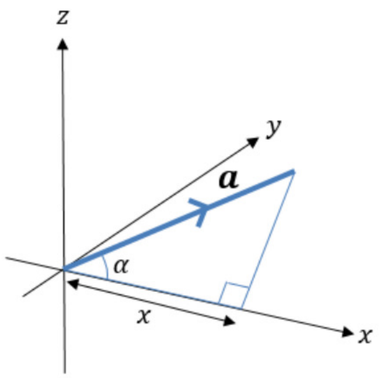
(3)

The point A' is a reflection of the point A in the plane Π

(b) Find, in terms of c, the coordinates of the point A'

(5)

Ratio of cosines of angles with each axis



Recall from Pure Year 2 that if a vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then from the diagram we can see that

$$\cos\alpha = \frac{x}{|\boldsymbol{a}|}$$

where α is the angle between $\boldsymbol{a}$ and the x-axis.

Similarly $\cos\beta = \frac{y}{|\boldsymbol{a}|}$ and $\cos\gamma = \frac{z}{|\boldsymbol{a}|}$

Thus $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \begin{pmatrix} \frac{x}{|\boldsymbol{a}|} \\ \frac{y}{|\boldsymbol{a}|} \\ \frac{z}{|\boldsymbol{a}|} \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}$

Recall that $\hat{\boldsymbol{a}}$ is the unit vector scaled version of $\boldsymbol{a}$.

Since $\hat{\boldsymbol{a}}$ is a unit vector, $|\hat{\boldsymbol{a}}| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$

✏ If α, β and γ are the angles between a vector $\boldsymbol{a}$ and each of the coordinate axes, then
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

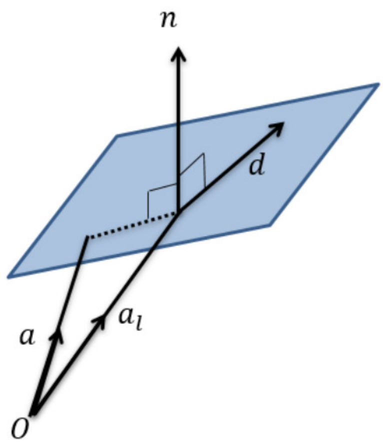
This is sometimes written as
$$l^2 + m^2 + n^2 = 1$$
Where l, m and n are the cosines of angles with each axis

A line L makes angles of α, β and γ with the x, y and z-axes respectively.
Show that $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.

Plane using vectors/lines on plane

Find, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = p$, an equation of the plane which contains the line l and the point with position vector a where l has the equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{a} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

Find also the Cartesian form of the equation.



Key strategy:

A. Identify two vectors on the plane.

B. n is cross product of these (as perpendicular to both)

C. Use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ as usual.

Find, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = p$, an equation of the plane which contains the line l and the point with position vector a where l has the equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\boldsymbol{a} = \begin{pmatrix} 9 \\ 8 \\ 5 \end{pmatrix}$

Find also the Cartesian form of the equation.

Vectors in direction of plane:

- $\boldsymbol{d} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$
- $\begin{pmatrix} 9 \\ 8 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}$

$$\boldsymbol{n} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \times \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -26 \\ 40 \\ -16 \end{pmatrix}$$

$\boldsymbol{a} \cdot \boldsymbol{n} = 6$

$\boldsymbol{r} \cdot \boldsymbol{n} = 6 \rightarrow -26x + 40y - 16z = 6$

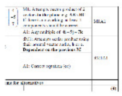
A plane Π passes through the points $A(2,2-1), B(3,2,-1), C(4,3,5)$
Find the equation of the plane Π in the form $\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$

Find the Cartesian equation of the same plane Π.

[June 2015 Q5] The points A, B and C have position vectors $\begin{pmatrix}1\\3\\2\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$ respectively.

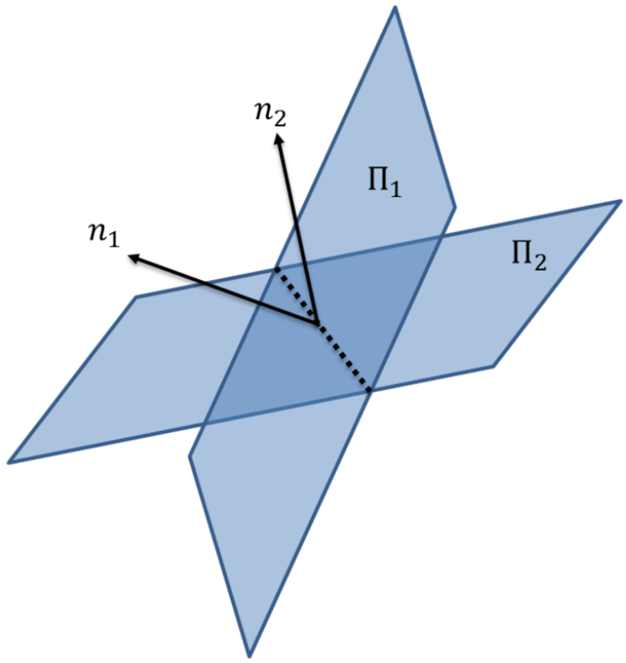
The plane Π contains the points A, B and C.

(c) Find a vector equation of Π in the form $\mathbf{r}.\mathbf{n} = p$. **(4)**



Intersecting Planes

Find the equation of the line of intersection of the planes Π_1 and Π_2 where Π_1 has equation $\boldsymbol{r} \cdot (2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 2$ and Π_2 has equation $\boldsymbol{r} \cdot (\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) = 5$.



Shortest Distance between Skew Lines

Find the shortest distance between the two skew lines with equations $\boldsymbol{r} = \boldsymbol{i} + \lambda(\boldsymbol{j} + \boldsymbol{k})$ and
$\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + \mu(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$, where λ and μ are scalars.

Shortest distance between 2 skew lines:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$$
$$\boldsymbol{r} = \boldsymbol{c} + \mu\boldsymbol{d}$$

$$= \left|\frac{(\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{b} \times \boldsymbol{d})}{|\boldsymbol{b} \times \boldsymbol{d}|}\right|$$

Test Your Understanding

[June 2009 Q7] The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} \alpha \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

...

For other values of α, the lines l_1 and l_2 do not intersect and are skew lines.

Given that $\alpha = 2$,

(*c*) find the shortest distance between the lines l_1 and l_2 .

(3)