Lower 6 Chapter 11

Vectors

Chapter Overview

1. Add/scale factors and show vectors are parallel.
2. Calculate magnitude and direction of a vector.
3. Understand and use position vectors.
4. Solve geometric problems.
5. Understand speed vs velocity.

9 **Vectors**	9.1	Use vectors in two dimensions.	Students should be familiar with column vectors and with the use of $\mathbf{i}$, and $\mathbf{j}$ unit vectors.
	9.2	Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.	Students should be able to find a unit vector in the direction of $\mathbf{a}$, and be familiar with the notation $\lvert\mathbf{a}\rvert$
	9.3	Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.	The triangle and parallelogram laws of addition. Parallel vectors.
	9.4	Understand and use position vectors; calculate the distance between two points represented by position vectors.	$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ The distance d between two points (x_1, y_1) and (x_2, y_2) is given by $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
	9.5	Use vectors to solve problems in pure mathematics and in context, (including forces).	For example, finding position vector of the fourth corner of a shape (e.g. parallelogram) $ABCD$ with three given position vectors for the corners A, B and C finding position vector of a point C on a line through A and B dividing AB in a given ratio, where position vectors of A and B are given. Contexts such as velocity, displacement, kinematics and forces will be covered in Paper 3, Sections 6.1, 7.3 and 8.1–8.4

Vector basics

Whereas a **coordinate** represents a **position** in space, a **vector** represents a **displacement** in space.

- A vector has 2 properties:
    - Direction
    - Magnitude (i.e. length)

If P and Q are points then $\overrightarrow{PQ}$ is the vector between them.

- If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, **they're the same vector** and are **parallel**.

- $\overrightarrow{AB} = -\overrightarrow{BA}$ and the two vectors are parallel, equal in magnitude but in **opposite directions**.

- Triangle Law for vector addition:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

The vector of multiple vectors is known as the **resultant vector.** (you will encounter this term in Mechanics)

- Vector **subtraction** is defined using vector addition and negation:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

- The zero vector $\mathbf{0}$ (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 2D**:** $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- A **scalar** is a normal number, which can be used to 'scale' a vector.
  - The **direction** will be the **same**.
  - But the **magnitude** will be **different** (unless the scalar is 1).

- Any vector parallel to the vector $\boldsymbol{a}$ can be written as $\lambda\boldsymbol{a}$, where λ is a scalar.

  The implication is that if we can write one vector **as a multiple of** another, then we can show they are parallel.

Example

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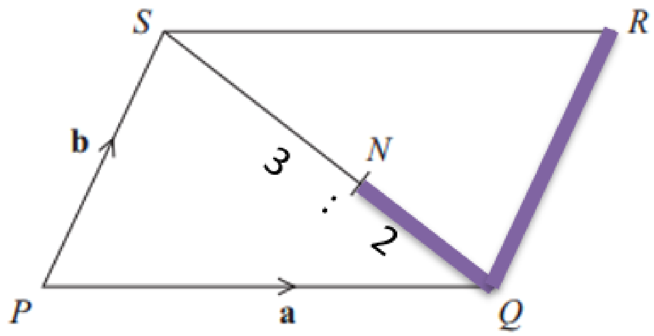

PQRS is a parallelogram.
N is the point on *SQ* such that $SN : NQ = 3 : 2$
$\overrightarrow{PQ} = \mathbf{a}$ $\overrightarrow{PS} = \mathbf{b}$

(a) Write down, in terms of **a** and **b**, an expression for $\overrightarrow{SQ}$.

(b) Express $\overrightarrow{NR}$ in terms of **a** and **b**.

Test your understanding

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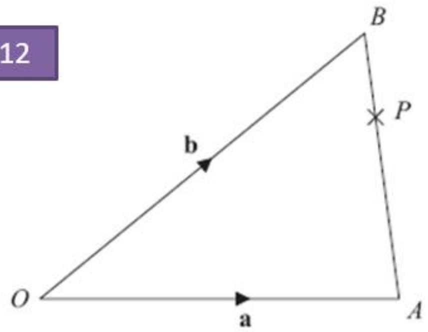


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = \mathbf{a}$
$\overrightarrow{OB} = \mathbf{b}$

(a) Find $\overrightarrow{AB}$ in terms of **a** and **b**.

....
(1)

P is the point on *AB* such that *AP* : *PB* = 3 : 1

(b) Find $\overrightarrow{OP}$ in terms of **a** and **b**.
Give your answer in its simplest form.

....

Exercise 11A Pg 234/235

Representing vectors

A **unit vector** is a vector of magnitude 1.

$\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the x-axis and y-axis respectively. We can write all vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$**.**

Example

If $\boldsymbol{a} = 3\boldsymbol{i}$, $\quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$, $\quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$ then:

1) Write $\boldsymbol{a}$ in vector form.
2) Find $\boldsymbol{b} + 2\boldsymbol{c}$ in $\boldsymbol{i}, \boldsymbol{j}$ form.

Exercise 11B Pg 237/238

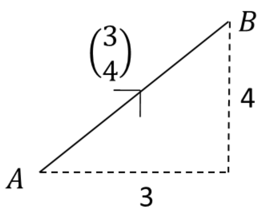
Magnitude of a vector

The magnitude $|a|$ of a vector $\boldsymbol{a}$ is its length.

$$\text{If } \boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix} \quad |\boldsymbol{a}| = \sqrt{x^2 + y^2}$$

Examples:

1.



2. $\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

3. $\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Direction of a Vector

The direction of a vector can be found using basic trigonometry.

Examples

1. Find the angle that vector $\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ makes with the positive x axis.

2. Find the angle that vector $\boldsymbol{b} = \begin{pmatrix} -5 \\ -12 \end{pmatrix}$ makes with $\boldsymbol{j}$.

Unit vector

A unit vector is a vector whose magnitude is 1.

If $\boldsymbol{a}$ is a vector, then the unit vector $\hat{\boldsymbol{a}}$ in the same direction is

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Example:

Find a unit vector in the direction of $\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Test Your Understanding: Convert the following vectors to unit vectors.

$\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Exercise 11C Pg 240/241

Position vectors

A vector used to represent a position is unsurprisingly known as a **position vector**.

A position can be thought of as a translation from the origin.

The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as $\boldsymbol{a}$.

Examples

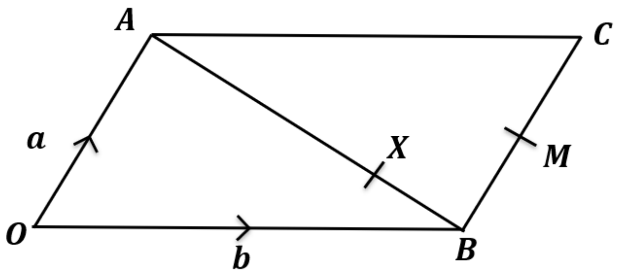
1. The points A and B have coordinates $(3,4)$ and $(11,2)$ respectively.
Find, in terms of i and j:

a) The position vector of A
b) The position vector of B
c) The vector $\overrightarrow{AB}$

2. $\overrightarrow{OA} = 5i - 2j$ and $\overrightarrow{AB} = 3i + 4j$. Find:
a) The position vector of B.
b) The exact value of $|\overrightarrow{OB}|$ in simplified surd form.

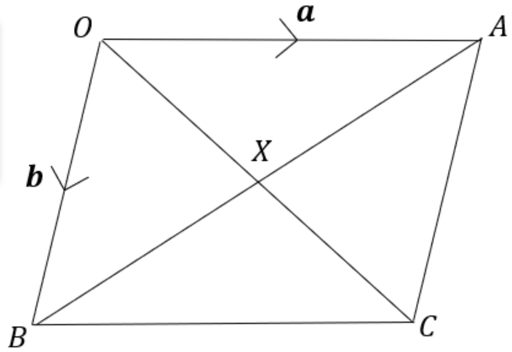
Exercise 11D Pg 243/244

Solving Geometric Problems

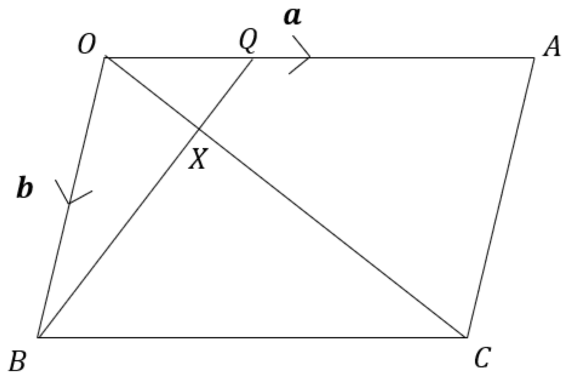


X is a point on AB such that $AX:XB = 3:1$. M is the midpoint of BC.
Show that $\overrightarrow{XM}$ is parallel to $\overrightarrow{OC}$.

$OACB$ is a parallelogram, where $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. The diagonals OC and AB intersect at a point X. Prove that the diagonals bisect each other.
(Hint: Perhaps find $\overrightarrow{OX}$ in two different ways?)



Test your understanding



In the above diagram, $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{3}\boldsymbol{a}$. We wish to find the ratio $OX: XC$.

a) If $\overrightarrow{OX} = \lambda\, \overrightarrow{OC}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and λ.
b) If $\overrightarrow{BX} = \mu\, \overrightarrow{BQ}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and μ.
c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX: XC$.

Area of a triangle example

If $\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{AC} = \boldsymbol{i} - 5\boldsymbol{j}$. Determine $\angle BAC$.

Extension

[STEP 2010 Q7]

Relative to a fixed origin O, the points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. (The points $O,\ A$ and B are not collinear.) The point C has position vector $\boldsymbol{c}$ given by

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b},$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\boldsymbol{p}$ and the lines OB and AC meet at the point Q with position vector $\boldsymbol{q}$. Show that

$$\boldsymbol{p} = \frac{\alpha a}{1 - \beta}$$

and write down q in terms of $\alpha,\ \beta$ and b.

Show further that the point R with position vector r given by

$$\boldsymbol{r} = \frac{\alpha\boldsymbol{a} + \beta\boldsymbol{b}}{\alpha + \beta},$$

lies on the lines OC and AB.

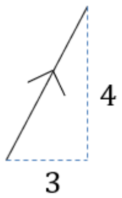
The lines OB and PR intersect at the point S. Prove that $\frac{OQ}{BQ} = \frac{OS}{BS}$.

Exercise 11E Pg 246/247

Modelling with vectors

In Mechanics, you will see certain things can be represented as a simple number (without direction), or as a vector (with direction):

Remember a 'scalar' just means a normal number (in the context of vectors). It can be obtained using the **magnitude** of the vector.

Vector Quantity	Equivalent Scalar Quantity
Velocity e.g. $\begin{pmatrix}3\\4\end{pmatrix}\ km/h$ This means the position vector of the object changes by $\begin{pmatrix}3\\4\end{pmatrix}$ each hour. 	
Displacement e.g. $\begin{pmatrix}-5\\12\end{pmatrix}\ km$	

Examples

1. A girl walks 2 km due east from a fixed point O to A, and then 3 km due south from A to B. Find

a) the total distance travelled

b) the position vector of B relative to O

c) $|\overrightarrow{OB}|$

d) The bearing of B from O.

2. In an orienteering exercise, a cadet leaves the starting point O and walks 15 km on a bearing of $120°$ to reach A, the first checkpoint. From A he walks 9 km on a bearing of $240°$ to the second checkpoint, at B. From B he returns directly to O.

Find:
a) the position vector of A relative to O
b) $|\overrightarrow{OB}|$
c) the bearing of B from O
d) the position vector of B relative O.

Exercise 11F Pg 250/251