9) Differentiation

9.1) Differentiating $\sin x$ and $\cos x$

Chapter CONTENTS

Worked example	Your turn
Prove, from first principles, that the derivative of $\sin x$ is $\cos x$. You may assume that as $h \to 0, \frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$	Prove, from first principles, that the derivative of $\sin x$ is $\cos x$. You may assume that as $h \to 0, \frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

Let $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \left(\left(\frac{\cos h - 1}{h} \right) \sin x + \left(\frac{\sin h}{h} \right) \cos x \right)$$
$$= \lim_{h\to 0} ((0)\sin x + (1)\cos x)$$
$$= \cos x$$

Worked example	Your turn
Find:	Find:
$$\frac{d}{dx}(\sin 5x)$$	$$\frac{d}{dx}(\sin\frac{1}{2}x)$$ $$\frac{1}{2}\cos 2x$$
$$\frac{d}{dx}(\cos\frac{1}{2}x)$$	$$\frac{d}{dx}(\cos 5x)$$ $$-5\sin 5x$$
$$\frac{d}{dx}(6\sin 7x - 4\cos 3x)$$	$$\frac{d}{dx}(7\sin 6x - 3\cos 4x)$$ $$42\cos 6x + 12\sin 4x$$

Worked example	Your turn
A curve has equation $y = \frac{1}{4}x - \cos 3x$. Find the stationary points on the curve in the interval $0 \le x \le \pi$	A curve has equation $y = \frac{1}{2}x - \cos 2x$. Find the stationary points on the curve in the interval $0 \le x \le \pi$ $(1.70, 1.82)$ and $(3.02, 0.539)$ (3 sf)

Worked example	Your turn
A curve has equation $y = \sin 5x + 3x$. Find the stationary points on the curve in the interval $0 \leq x \leq \frac{3}{5}\pi$	A curve has equation $y = \sin 3x + 2x$. Find the stationary points on the curve in the interval $0 \leq x \leq \frac{2}{3}\pi$ $(0.767, 2.279)$ and $(1.328, 1.910)$ (3 sf)

Worked example	Your turn
A curve has equation $y = \sin 4x - \cos 3x$. Find the equation of the tangent to the curve at the point $(\pi, 1)$	A curve has equation $y = \sin 4x + \cos 3x$. Find the equation of the tangent to the curve at the point $(\pi, -1)$ $y = 4x - 4\pi - 1$

Worked example	Your turn
A curve has equation $y = 3x^2 - \sin x$. Show that the equation of the normal to the curve at the point with x-coordinate π is $x + (6\pi + 1)y - \pi(18\pi^2 - 3\pi - 1) = 0$	A curve has equation $y = 3x^2 + \sin x$. Show that the equation of the normal to the curve at the point with x-coordinate π is $x + (6\pi - 1)y - \pi(18\pi^2 - 3\pi + 1) = 0$ Shown

9.2) Differentiating exponentials and logarithms

Worked example	Your turn
Prove that the derivative of a^{kx} is $a^{kx} k \ln a$	Prove that the derivative of a^{x} is $a^{x} \ln a$ Proof

Worked example	Your turn
Find: $$\frac{d}{dx}(5e^{2x})$$ $$\frac{d}{dx}(4e^{\frac{1}{3}x})$$ $$\frac{d}{dx}(3e^{-0.5x})$$	Find: $$\frac{d}{dx}(2e^{\frac{1}{4}x})$$ $$\frac{1}{2}e^{\frac{1}{4}x}$$

Worked example	Your turn
Find:	Find:
$$\frac{d}{dx}(2^x)$$	$$\frac{d}{dx}(5^x)$$
	$$5^x \ln 5$$
$$\frac{d}{dx}(3^x)$$	
$$\frac{d}{dx}(4^x)$$	

Worked example	Your turn
Find: $$\frac{d}{dx}(5(2^x))$$ $$\frac{d}{dx}(4(3^x))$$ $$\frac{d}{dx}(3(4^x))$$	Find: $$\frac{d}{dx}(2(5^x))$$ $$5^x\ 2\ln 5$$

Worked example	Your turn
Find: $\frac{d}{dx}(2^{5x})$ $\frac{d}{dx}(3^{4x})$ $\frac{d}{dx}(4^{3x})$	Find: $\frac{d}{dx}(5^{2x})$ $5^{2x}\,2\ln 5$

Worked example	Your turn
Find: $$\frac{d}{dx}(3^x)$$ $$\frac{d}{dx}(x^3)$$	Find: $$\frac{d}{dx}(5^x)$$ $$5^x \ln 5$$ $$\frac{d}{dx}(x^5)$$ $$5x^4$$

Worked example	Your turn
Find:	Find:
$$\frac{d}{dx}(\ln 2x)$$	$$\frac{d}{dx}(\ln 5x)$$
	$$\frac{1}{x}$$
$$\frac{d}{dx}(\ln 3x)$$	
$$\frac{d}{dx}(\ln 4x)$$	

Worked example	Your turn
Find:	Find:
$$\frac{d}{dx}(2\ln x)$$	$$\frac{d}{dx}(5\ln x)$$
	$$\frac{5}{x}$$
$$\frac{d}{dx}(3\ln x)$$	
$$\frac{d}{dx}(4\ln x)$$	

Worked example	Your turn
Find: $$\frac{d}{dx}(\ln x^2)$$ $$\frac{d}{dx}(\ln x^3)$$ $$\frac{d}{dx}(\ln x^4)$$	Find: $$\frac{d}{dx}(\ln x^5)$$ $$\frac{5}{x}$$

Worked example	Your turn
Find: $$\frac{d}{dx}\left(\frac{3+2e^{5x}}{7e^{4x}}\right)$$ $$\frac{d}{dx}\left(\frac{7-5e^{-4x}}{3e^{2x}}\right)$$	Find: $$\frac{d}{dx}\left(\frac{2-3e^{7x}}{4e^{3x}}\right)$$ $$-\frac{3}{2}e^{-3x}-3e^{4x}$$

Worked example	Your turn
Find: $$\frac{d}{dx}(e^x - 3)^2$$	Find: $$\frac{d}{dx}(e^x + 5)^2$$ $$2e^{2x} + 10e^x$$

Worked example	Your turn
The population P of a species after t days can be modelled using $P = 640(2^{-3t})$ a) Determine how many days have elapsed before there are 20 individuals left b) Determine the rate of change of the population after 3 days.	The population P of a species after t days can be modelled using $P = 460(3^{-2t})$ a) Determine how many days have elapsed before there are 20 individuals left b) Determine the rate of change of the population after 3 days. a) 1.43 days b) -1.39 individuals/day

Worked example	Your turn
A curve has the equation $$y = e^{4x} - \ln x$$ Show that the equation of the tangent to the curve at the point with x-coordinate 1 is: $$y = (4e^4 - 1)x - 3e^4 + 1$$	A curve has the equation $$y = e^{3x} - \ln x$$ Show that the equation of the tangent to the curve at the point with x-coordinate 1 is: $$y = (3e^3 - 1)x - 2e^3 + 1$$ Shown

Worked example	Your turn
A curve has the equation $$y = 5(2^{3x})$$ Find the equation of the normal to the curve at the point with x-coordinate 1 in the form $y = ax + b$, where a and b are constants to be found in exact form	A curve has the equation $$y = 4(3^{2x})$$ Find the equation of the normal to the curve at the point with x-coordinate 1 in the form $y = ax + b$, where a and b are constants to be found in exact form

$$y = -\frac{1}{72\ln 3}x + \frac{1}{72\ln 3} + 36$$

$$a = -\frac{1}{72\ln 3}$$

$$b = \frac{1}{72\ln 3} + 36$$

9.3) The chain rule

Worked example	Your turn
Differentiate with respect to x: $y = (4x^3 - x)^7$ $f(x) = (7x^2 - 3x)^4$	Differentiate with respect to x: $y = (3x^4 + x)^5$ $\frac{dy}{dx} = 5(3x^4 + x)^4(12x^3 + 1)$

Worked example	Your turn
Differentiate with respect to x: $y = (\ln x)^3$ $f(x) = (\ln x)^4$	Differentiate with respect to x: $y = (\ln x)^5$ $\frac{dy}{dx} = \frac{5(\ln x)^4}{x}$

Worked example	Your turn
Differentiate with respect to x: $$y = e^{x^2+x}$$ $$f(x) = e^{x^3-2x+1}$$	Differentiate with respect to x: $$y = e^{x^4-3x^2-1}$$ $$\frac{dy}{dx} = (4x^3 - 6x)e^{x^4-3x^2-1}$$

Worked example	Your turn
Differentiate with respect to x: $y=(3^x+1)^2$ $f(x)=(2^x-3x)^4$	Differentiate with respect to x: $y=(4^x-2x)^3$ $\frac{dy}{dx}=3(4^x-2x)^2(4^x\ln 4-2)$

Worked example	Your turn
Differentiate with respect to x: $$y = \ln(\sin x)$$	Differentiate with respect to x: $$y = \ln(\cos x)$$ $$\frac{dy}{dx} = \tan x$$

Worked example	Your turn
Differentiate with respect to x: $y = \ln(x^2 - 3x + 4)$	Differentiate with respect to x: $y = \ln(x^4 - 5x^2 - 3)$
	$\frac{dy}{dx} = \frac{4x^3 - 10x}{x^4 - 5x^2 - 3}$
$f(x) = \ln(x^3 - 2x - 5)$	

Worked example	Your turn
Differentiate with respect to x: $y = \sin 5x$ $f(x) = \cos(-4x)$	Differentiate with respect to x: $y = \cos 3x$ $\frac{dy}{dx} = -3\sin 3x$

Worked example	Your turn
Differentiate with respect to x: $$y = \sin^4 x$$ $$f(x) = \cos^3 x$$	Differentiate with respect to x: $$y = \sin^2 x$$ $$\frac{dy}{dx} = 2\sin x \cos x = \sin 2x$$

Worked example	Your turn
Differentiate with respect to x: $y = \sin^4 3x$ $f(x) = \cos^3 2x$	Differentiate with respect to x: $y = \sin^2 3x$ $\frac{dy}{dx} = 6 \sin 3x \cos 3x = 3 \sin 6x$

Worked example	Your turn
Differentiate with respect to x: $y = \sqrt{2x - 1}$ $f(x) = \sqrt[3]{4x^2 + 5}$	Differentiate with respect to x: $y = \sqrt[4]{2x^3 + 1}$ $\frac{dy}{dx} = \frac{1}{4}(2x^3 + 1)^{-\frac{3}{4}}(6x^2)$

Worked example	Your turn
Differentiate with respect to x: $$y = e^{e^{-x}}$$	Differentiate with respect to x: $$y = e^{e^{x}}$$ $$\frac{dy}{dx} = e^{x}\left(e^{e^{x}}\right)$$

Worked example	Your turn
Given that $y = \sqrt{2x^5 - 2}$, find $\frac{dy}{dx}$ at $(3,22)$	Given that $y = \sqrt{5x^2 + 1}$, find $\frac{dy}{dx}$ at $(4,9)$ $\frac{20}{9}$

Worked example	Your turn
Find the gradient of: $x = (1+2y)^3$ when $y = 1$ $x = (3y^2 - 2)^4$ at $(1, -1)$	Find the gradient of: $x = (2y^3 - 3)^5$ at $(-1,1)$ 30

Worked example	Your turn
Find $\frac{dy}{dx}$ when:	Find $\frac{dy}{dx}$ when:
$x = 2y^2 + y$	$x = 4y^3 - y + 2$
	$\frac{1}{12y^2 - 1}$
$x = 3y^4 - y^2 - 1$	

Worked example	Your turn
A curve C has equation $$y = \frac{3}{(2-5x)^4}, x \neq \frac{2}{5}$$ Find an equation for the normal to C at the point with x-coordinate 1 in the form $ax + by + c = 0$, where a, b and c are integers	A curve C has equation $$y = \frac{4}{(3-2x)^5}, x \neq \frac{3}{2}$$ Find an equation for the normal to C at the point with x-coordinate 2 in the form $ax + by + c = 0$, where a, b and c are integers $x + 40y + 158 = 0$

9.4) The product rule

Worked example	Your turn
Differentiate with respect to x: $y = x^2e^{3x}$ $f(x) = x^5e^{2x}$	Differentiate with respect to x: $y = x^3e^{2x}$ $\frac{dy}{dx} = x^2e^{2x}(2x + 3)$

Worked example	Your turn
Differentiate with respect to x: $y = x^2 \ln 3x$ $f(x) = x^4 \ln 5x$	Differentiate with respect to x: $y = x^3 \ln 2x$ $\frac{dy}{dx} = x^2(1 + 3\ln 2x)$

Worked example	Your turn
Differentiate with respect to x: $y = 2x^3(4x-5)^6$	Differentiate with respect to x: $y = 3x^2(6x-5)^4$
	$\frac{dy}{dx} = 6x(6x-5)^3(18x-5)$
$f(x) = 6x^5(4x-3)^2$	

Worked example	Your turn
Differentiate with respect to x: $y = x^3 \sin x$	Differentiate with respect to x: $y = x^2 \sin x$
	$\frac{dy}{dx} = x(x\cos x + 2\sin x)$
$f(x) = x^4 \cos x$	

Worked example	Your turn
Differentiate with respect to x: $$y = e^{3x}\sin^4 2x$$ $$f(x) = e^{2x}\cos^3 4x$$	Differentiate with respect to x: $$y = e^{4x}\sin^2 3x$$ $$\frac{dy}{dx} = e^{4x}\sin 3x\,(6\cos 3x + 4\sin 3x)$$

Worked example	Your turn
Determine the coordinates of the turning point: $y = xe^{3x}$ $f(x) = xe^{4x}$	Determine the coordinates of the turning point: $y = xe^{2x}$ $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$

Worked example	Your turn
Differentiate with respect to x: $$y = x^2\sqrt{5x-2}$$	Differentiate with respect to x: $$y = x^2\sqrt{3x-1}$$ $$\frac{dy}{dx} = \frac{x(15x-4)}{2\sqrt{3x-1}}$$

Worked example	Your turn
Find the equation of the tangent to the curve with equation $y = x^2\sin(x^2)$ at the point $\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a, b and c are exact constants.	Find the equation of the tangent to the curve with equation $y = x^2\cos(x^2)$ at the point $\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a, b and c are exact constants. $\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2}\left(\frac{\pi - 2}{2}\right) = 0$

Worked example	Your turn
Differentiate with respect to x: $$\frac{dy}{dx} = \frac{1}{5x(x+1)^{\frac{1}{2}}}$$	Differentiate with respect to x: $$\frac{dy}{dx} = \frac{1}{6x(x-1)^{\frac{1}{2}}}$$ $$\frac{d^2y}{dx^2} = \frac{2-3x}{12x^2(x-1)^{\frac{3}{2}}}$$

9.5) The quotient rule

Chapter CONTENTS

Worked example	Your turn
Differentiate with respect to x: $$y = \frac{3x}{5x-2}$$	Differentiate with respect to x: $$y = \frac{x}{2x+5}$$ $$\frac{dy}{dx} = \frac{5}{(2x+5)^2}$$

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$y = \frac{x^2}{\ln 5x}$	$y = \frac{x^3}{\ln 3x}$
	$\frac{dy}{dx} = \frac{x^2(3\ln 3x - 1)}{(\ln 3x)^2}$
$y = \frac{\ln 4x}{x^4}$	

Worked example	Your turn
Differentiate with respect to x: $$y = \frac{\cos 3x}{x^4}$$ $$f(x) = \frac{x^4}{\cos 3x}$$	Differentiate with respect to x: $$y = \frac{\sin 4x}{x^3}$$ $$\frac{dy}{dx} = \frac{4x\cos 4x - 3\sin 4x}{x^4}$$

Worked example	Your turn
Find the stationary point of $y = \frac{\cos x}{e^{3x}}, 0 < x < \pi$	Find the stationary point of $y = \frac{\sin x}{e^{2x}}, 0 < x < \pi$ $(0.464, 0.177)$ (3 dp)

Worked example	Your turn
Find the equation of the tangent to the curve $y = \frac{e^{\frac{1}{4}x}}{x}$ at the point $(4, \frac{1}{4}e)$	Find the equation of the tangent to the curve $y = \frac{e^{\frac{1}{2}x}}{x}$ at the point $(2, \frac{1}{2}e)$ $y = \frac{1}{2}e$

9.6) Differentiating trigonometric functions

Chapter CONTENTS

Worked example	Your turn
Differentiate with respect to x: $$y = \cot x$$	Differentiate with respect to x: $$y = \tan x$$ $$\frac{dy}{dx} = \sec^2 x$$

Worked example	Your turn
Differentiate with respect to x: $$y = \operatorname{cosec} x$$	Differentiate with respect to x: $$y = \sec x$$ $$\frac{dy}{dx} = \sec x \tan x$$

Worked example	Your turn
Differentiate with respect to x: $y = \tan 2x$ $f(x) = \tan(-\frac{x}{3})$	Differentiate with respect to x: $y = \tan 4x$ $\frac{dy}{dx} = 4\sec^2 4x$

Worked example	Your turn
Differentiate with respect to x: $y = \cot 2x$	Differentiate with respect to x: $y = \cot 4x$

$$\frac{dy}{dx} = -4\ cosec^2 4x$$

$$f(x) = \cot(-\frac{x}{3})$$

Worked example	Your turn
Differentiate with respect to x: $$y = \sec 2x$$ $$f(x) = \sec(-\frac{x}{3})$$	Differentiate with respect to x: $$y = \sec 4x$$ $$\frac{dy}{dx} = 4\sec 4x \tan 4x$$

Worked example	Your turn
Differentiate with respect to x: $$y = \operatorname{cosec} 2x$$ $$f(x) = \operatorname{cosec}(-\frac{x}{3})$$	Differentiate with respect to x: $$y = \operatorname{cosec} 4x$$ $$\frac{dy}{dx} = -4\, cosec\, 4x \cot 4x$$

Worked example	Your turn
Differentiate with respect to x: $$y = \tan^4 2x$$ $$f(x) = \tan^3\left(-\frac{x}{3}\right)$$	Differentiate with respect to x: $$y = \tan^2 4x$$ $$\frac{dy}{dx} = 8\tan 4x \sec^2 4x$$

Worked example	Your turn
Differentiate with respect to x: $y = \cot^4 2x$ $f(x) = \cot^3(-\frac{x}{3})$	Differentiate with respect to x: $y = \cot^2 4x$ $\frac{dy}{dx} = -8\cot 4x\, cosec^2 4x$

Worked example	Your turn
Differentiate with respect to x: $y = \sec^4 2x$ $f(x) = \sec^3(-\frac{x}{3})$	Differentiate with respect to x: $y = \sec^2 4x$ $\frac{dy}{dx} = 8\sec^2 4x \tan 4x$

Worked example	Your turn
Differentiate with respect to x: $y = \operatorname{cosec}^4 2x$ $f(x) = \operatorname{cosec}^3(-\frac{x}{3})$	Differentiate with respect to x: $y = \operatorname{cosec}^2 4x$ $\frac{dy}{dx} = -8\, cosec^2\, 4x \cot 4x$

Worked example	Your turn
Differentiate with respect to x: $$y = \frac{cosec\ 3x}{x^3}$$	Differentiate with respect to x: $$y = \frac{cosec\ 2x}{x^2}$$ $$\frac{dy}{dx} = -\frac{2cosec\ 2x(x\cot 2x + 1)}{x^3}$$

Worked example	Your turn
Given that $x = \cot y$, express $\frac{dy}{dx}$ in terms of x.	Given that $x = \tan y$, express $\frac{dy}{dx}$ in terms of x. $$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

Worked example	Your turn
Differentiate with respect to x: $y = \arccos x$ $y = \arctan x$	Differentiate with respect to x: $y = \arcsin x$ $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

Worked example	Your turn
Differentiate with respect to x: $y = \arccos x^4$ $y = \arctan x^3$	Differentiate with respect to x: $y = \arcsin x^2$ $\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^4}}$

Worked example	Your turn
Given that $x = \operatorname{cosec} 3y$, find $\frac{dy}{dx}$ in terms of x	Given that $x = \sec 2y$, find $\frac{dy}{dx}$ in terms of x $$\frac{dy}{dx} = \frac{1}{2x\sqrt{x^2 - 1}}$$

Worked example	Your turn
Given that $y = \arctan\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$	Given that $y = \arctan\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$ $$\frac{dy}{dx} = -\frac{1}{1+x^2}$$

9.7) Parametric differentiation

Chapter CONTENTS

Worked example	Your turn
Find the gradient at the point P where $t = 3$, on the curve given parametrically by $x = t^2 - t, \quad y = \mathrm{t}^4 - 2, \quad \mathrm{t} \in \mathbb{R}$	Find the gradient at the point P where $t = 2$, on the curve given parametrically by $x = t^3 + t, \quad y = \mathrm{t}^2 + 1, \quad \mathrm{t} \in \mathbb{R}$ $\frac{4}{13}$

Worked example

Find the equation of the tangent at the point where $t = \frac{\pi}{6}$, to the curve with parametric equations

$$x = \sqrt{5}\sin 2t\,, \qquad y = 8\cos^2 t\,, \qquad 0 \le t \le \pi$$

Your turn

Find the equation of the tangent at the point where $t = \frac{\pi}{3}$, to the curve with parametric equations

$$x = \sqrt{3}\sin 2t\,, \qquad y = 4\cos^2 t\,, \qquad 0 \le t \le \pi$$

$$y = 2x - 2$$

Worked example	Your turn
Find the equation of the normal at the point where $\theta = \frac{\pi}{3}$, to the curve with parametric equations $$x = 2\cos\theta\,, \qquad y = 7\sin\theta$$	Find the equation of the normal at the point where $\theta = \frac{\pi}{6}$, to the curve with parametric equations $$x = 3\sin\theta\,, \qquad y = 5\cos\theta$$ $$y = \frac{3\sqrt{3}}{5}x + \frac{8\sqrt{3}}{5}$$

9.8) Implicit differentiation

Chapter CONTENTS

Worked example	Your turn
Find:	Find:
$$\frac{d}{dx}(y^4)$$	$$\frac{d}{dx}(2y^3)$$
	$$6y^2\frac{dy}{dx}$$
$$\frac{d}{dx}(3y^5)$$	

Worked example	Your turn
Find: $$\frac{d}{dx}(\cos y)$$ $$\frac{d}{dx}(\tan 2y)$$	Find: $$\frac{d}{dx}(\sin 3y)$$ $$3\cos 3y\frac{dy}{dx}$$

Worked example	Your turn
Find: $$\frac{d}{dx}(e^y)$$ $$\frac{d}{dx}(e^{2y})$$	Find: $$\frac{d}{dx}(e^{3y})$$ $$3e^{3y}\frac{dy}{dx}$$

Worked example	Your turn
Find: $$\frac{d}{dx}(xy)$$ $$\frac{d}{dx}(x^2y)$$	Find: $$\frac{d}{dx}(x^3y)$$ $$x^3\frac{dy}{dx} + 3x^2y$$

Worked example	Your turn
Find: $$\frac{d}{dx}(e^{xy})$$ $$\frac{d}{dx}(e^{x^2y})$$	Find: $$\frac{d}{dx}(e^{x^3y})$$ $$\left(x^3\frac{dy}{dx}+3x^2y\right)e^{x^3y}$$

Worked example	Your turn
Find: $$\frac{d}{dx}(\cos(x+y))$$ $$\frac{d}{dx}(\tan(x^2-4y))$$	Find: $$\frac{d}{dx}(\sin(x^3+5y))$$ $$\left(3x^2+5\frac{dy}{dx}\right)(\cos(x^3+5y))$$

Worked example	Your turn
Find $\frac{dy}{dx}$ where: $x^4 - x + y^2 - 3y = 5$	Find $\frac{dy}{dx}$ where: $x^3 + x + y^3 + 3y = 6$ $\frac{dy}{dx} = \frac{-3x^2 - 1}{3y^2 + 3}$

Worked example	Your turn
Find $\frac{dy}{dx}$ at the point $(1, 1)$, given that: $$6x^2y - \frac{4x}{y^2} = 2$$	Find $\frac{dy}{dx}$ at the point $(1, 1)$, given that: $$4xy^2 + \frac{6x^2}{y} = 10$$ $$\frac{dy}{dx} = -8$$

Worked example	Your turn
A curve is described by: $$x^3 + 4y^2 = -12xy$$ Find the gradient of the curve at the points where $x = 8$	A curve is described by: $$x^3 - 4y^2 = 12xy$$ Find the gradient of the curve at the points where $x = -8$ $$\frac{dy}{dx} = -3 \text{ at } (-8, 16)$$ $$\frac{dy}{dx} = 0 \text{ at } (-8, 8)$$

Worked example	Your turn
$x^2 + y^2 + 20x + 4y - 8xy = -75$ Find the values of y for which $\frac{dy}{dx} = 0$	$x^2 + y^2 + 10x + 2y - 4xy = 10$ Find the values of y for which $\frac{dy}{dx} = 0$ $y = \frac{7}{3}, 5$

Worked example	Your turn
A curve has equation $$x^2 + 4xy + y^2 - x = 35$$ Find the equation of the tangent to the curve at the point $(2, 3)$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers	A curve has equation $$x^2 + 2xy - y^2 + x = 20$$ Find the equation of the tangent to the curve at the point $(3, 2)$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers $$11x + 2y - 37 = 0$$

Worked example	Your turn
The curve $ye^{-4x} = 4x - y^2$. Find the equation of the normal(s) to the curve at the point where $x = 0$. Give your answer in the form $ax + by + c = 0$	The curve $ye^{-2x} = 2x + y^2$. Find the equation of the normal(s) to the curve at the point where $x = 0$. Give your answer in the form $ax + by + c = 0$ $2x + y = 0$ at $(0,0)$ $x - 4y + 4 = 0$ at $(0, 1)$

9.9) Using second derivatives

Worked example	Your turn
Find the interval on which the function is concave: $$f(x) = x^3 - 2x + 5$$ $$g(x) = 2x^3 - 5x^2 - 6$$	Find the interval on which the function is concave: $$h(x) = x^3 + 4x + 3$$ $$x \leq 0$$

Worked example	Your turn
Find the interval on which the function is convex: $$f(x) = x^3 - 2x + 5$$ $$g(x) = 2x^3 - 5x^2 - 6$$	Find the interval on which the function is convex: $$h(x) = x^3 + 4x + 3$$ $$x \geq 0$$

Worked example	Your turn
Show that the function is convex for all real values of x: $$f(x) = e^{3x} + x^2$$ $$g(x) = e^{4x} + x^4$$	Show that the function is convex for all real values of x: $$h(x) = e^{2x} + x^2$$ Shown that $h''(x) \geq 0$ for $x \in \mathbb{R}$

Worked example	Your turn
Determine if there is a point of inflection on the curve with equation $y = (x-3)^4$	Determine if there is a point of inflection on the curve with equation $y = (x+5)^4$

No

$$\frac{d^2y}{dx^2} = 12(x+5)^2 \geq 0$$

Either side of $x = -5$, $\frac{d^2y}{dx^2}$ always positive.

Therefore local minimum at $(-5, 0)$

Worked example	Your turn
A curve C has equation $$y = \frac{1}{4}x^2 \ln x - 3x + 7, x > 0$$ Find where C is convex	A curve C has equation $$y = \frac{1}{5}x^2 \ln x + 4x - 3, x > 0$$ Find where C is convex $$x \geq e^{-\frac{3}{2}}$$

9.10) Rates of change

Worked example	Your turn
Atmospheric pressure decreases as altitude increases. The rate at which atmospheric pressure decreases is proportional to the current air pressure. Write down a differential equation for the rate of change of the atmospheric pressure.	In the decay of radioactive particles, the rate at which particles decay is proportional to the number of particles remaining. Write down a differential equation for the rate of change of the number of particles. N = number of particles t = time $\frac{dN}{dt} = -kN, k > 0$

Worked example	Your turn
An metal bar is heated from a starting temperature. It is noted that the rate of gain of temperature of the metal bar is proportional to the difference between the temperature of the metal bar and its surroundings. Write a differential equation to represent this situation.	An object is heated to a certain temperature, then allowed to cool down. It is noted that the rate of loss of temperature of the object is proportional to the difference between the temperature of the object and its surroundings. Write a differential equation to represent this situation. θ = temperature of object (degrees) t = time (seconds) θ_0 = temperature of the surroundings $\frac{d\theta}{dt} = -k(\theta - \theta_0), k > 0$ Newton's law of cooling

Worked example	Your turn
Determine the rate of change of the area A of a circle when the radius $r = 7$cm, given that the radius is changing at a rate of $4\ cm\ s^{-1}$.	Determine the rate of change of the area A of a circle when the radius $r = 3$cm, given that the radius is changing at a rate of $5\ cm\ s^{-1}$. $\frac{dA}{dt} = 30\pi\ cm^2\ s^{-1}$

Worked example	Your turn
A cube is expanding uniformly as it is heated. At time t seconds, the length of each edge is x cm and the volume of the cube is $V\ cm^3$. Given that the volume increases at a constant rate of $0.024\ cm^3\ s^{-1}$, find the rate of increase of the total surface area of the cube in $cm^2\ s^{-1}$ when $x = 4$	A cube is expanding uniformly as it is heated. At time t seconds, the length of each edge is x cm and the volume of the cube is $V\ cm^3$. Given that the volume increases at a constant rate of $0.048\ cm^3\ s^{-1}$, find the rate of increase of the total surface area of the cube in $cm^2\ s^{-1}$ when $x = 8$ $0.024\ cm^2\ s^{-1}$

Worked example	Your turn
A right circular cylindrical metal rod is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm. The cross-sectional area of the rod is increasing at the constant rate of $0.016\ cm^2\ s^{-1}$. Find the rate of increase of the volume of the rod when $x = 4$	A right circular cylindrical metal rod is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm. The cross-sectional area of the rod is increasing at the constant rate of $0.032\ cm^2\ s^{-1}$. Find the rate of increase of the volume of the rod when $x = 2$ $0.48\ cm^3\ s^{-1}$