## Forced Harmonic Motion

In addition to the 'natural' forces acting on the particle, i.e. damping force and restoring force, there may be a further a further force acting on the particle. This is known as **forced harmonic motion**.

All structures have natural frequencies of vibration. If an external agent causes them to vibrate at or close to one of these frequencies it can create resonance which can have devastating effects. Engineers must be able to predict these natural frequencies.

Forced harmonic motion

$$\frac{dx^2}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

We can solve problems like this using the Non-homogeneous DE method.

Example

A particle $P$ of mass 1.5 kg is moving on the $x$-axis. At time $t$ the displacement of $P$ from the origin $O$ is $x$ metres and the speed of $P$ is $v$ ms$^{-1}$. Three forces act on $P$, namely a restoring force of magnitude $7.5x$ N, a resistance to the motion of $P$ of magnitude $6v$ N and a force of magnitude $12 \sin t$ N acting in the direction $OP$. When $t = 0,\ x = 5$ and $\frac{dx}{dt} = 2$.

(a) Show that $\frac{dx^2}{dt^2} + 4\frac{dx}{dt} + 5x = 8 \sin t$

(b) Find $x$ as a function of $t$.

Describe the motion when $t$ is large.

2. A particle $P$ is attached to end $A$ of a light elastic spring $AB$. Initially the particle and the string lie at rest on a smooth horizontal plane. At time $t = 0$, the end $B$ of the string is set in motion and moves with constant speed $U$ in the direction $AB$, and the displacement of $P$ from $A$ is $x$. Air resistance acting on $P$ is proportional to its speed. The subsequent motion can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + k^2x = 2kU$$

Find an expression for $x$ in terms of $U,\ k$ and $t$