8A Modelling with First Order Differentials

1. A particle P starts from rest at a given point O and moves along a straight line. At time t seconds, the acceleration, $a\ ms^{-2}$, of P is given by:

$$a = \frac{6}{(t-2)^2}, t \geq 0$$

a) Find the velocity of P at time t seconds

b) Show that the displacement of P from O when $t = 6$ is given by $(18 - 12ln2)\ m$

2. A particle P is travelling along a straight line. At time t seconds, the acceleration of the particle is given by:

$$a = t + \frac{3}{t}v,\ t \geq 0$$

Given that $v = 0$ when $t = 2$, show that the velocity of the particle at time t is given by the equation:

$$v = ct^3 - t^2$$

where c is a constant to be found.

3. A storage tank initially contains 1000 litres of pure water. Liquid is removed from the tank at a constant rate of 30 litres per hour and a chemical solution is added at a constant rate of 40 litres per hour. The chemical solution contains 4 grams of copper sulphate per litre of water.

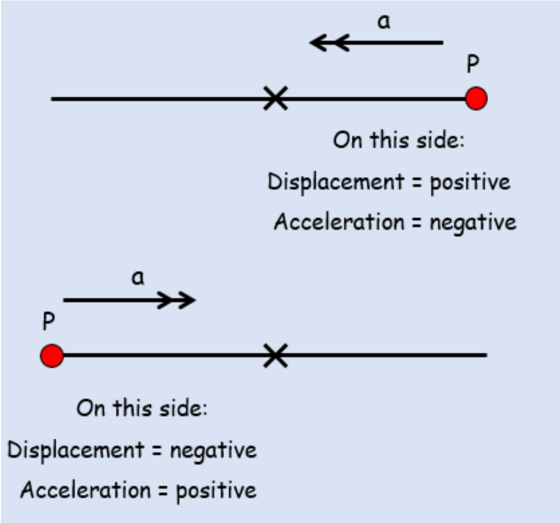
a) Given that there are x grams of copper sulphate in the tank after t hours and that the copper sulphate immediately disperses throughout the tank upon entry, show that the situation can be modelled by the differential equation:

$$\frac{dx}{dt} = 160 - \frac{3x}{100 + t}, \quad t \geq 0$$

b) Hence, find the number of grams of copper sulphate in the tank after 6 hours.

c) Suggest a possible refinement for the model

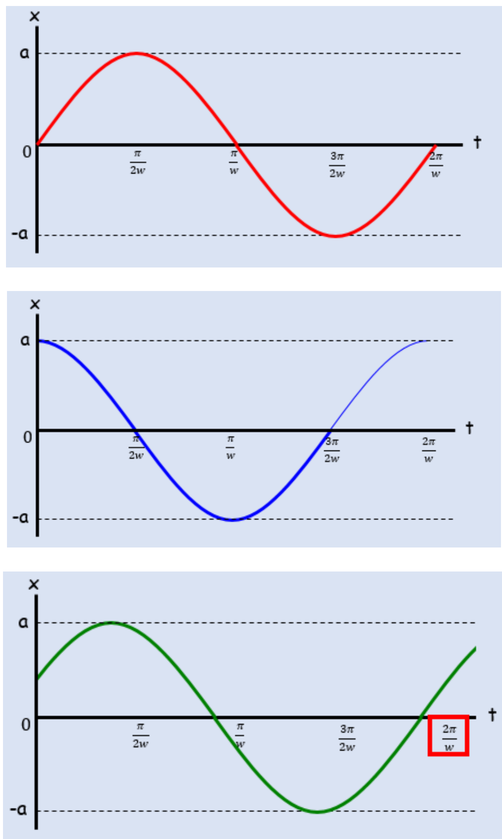
8B Simple Harmonic Motion



$$\ddot{x} = -w^2 x$$

$$v^2 = w^2(a^2 - x^2)$$

$$x = a\sin(wt + C)$$



$$Period = \frac{2\pi}{w}$$

1. A particle is moving along a straight line. At time t seconds its displacement, $x\ m$ from a fixed point O is such that:

$$\frac{d^2x}{dt^2} = -4x$$

Given that at $t = 0$, $x = 1$ and that the particle is moving with velocity $4\ ms^{-1}$:

a) Find an expression for the particle's displacement after t seconds

b) Determine the maximum displacement of the particle from O.

2. A particle P is attached to the ends of two identical elastic springs. The free ends of the springs are attached to two points A and B. The point C lies between A and B such that ABC is a straight line and $AB \neq BC$. The particle is held at C and then released from rest.
At time t seconds, the displacement of the particle from C is $x\,m$ and its velocity is $v\,ms^{-1}$.
The subsequent motion can be described by the differential equation $\ddot{x} = -25x$.

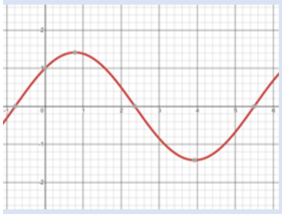
a) Describe the motion of the particle

b) Given that when $t = 0$, $x = 0.4$ and $v = 0$, find x as a function of t

c) State the period of the motion and state the maximum speed of P.

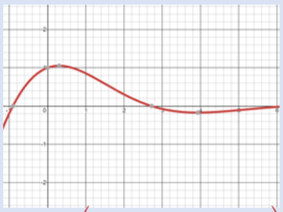
8C Part 1 Damped Harmonic Motion

$\frac{d^2x}{dt^2} + x = 0$



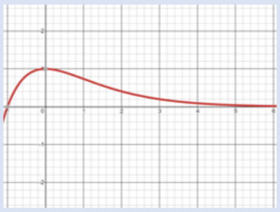
$\boldsymbol{x = cost + sint}$

$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$



$\boldsymbol{x = e^{-\frac{1}{2}t}\left(cos\left(\frac{\sqrt{3}}{2}t\right) + sin\left(\frac{\sqrt{3}}{2}t\right)\right)}$

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$



$\boldsymbol{x = (1+t)e^{-t}}$

$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0$

$\boldsymbol{x = e^{\left(\frac{-3+\sqrt{5}}{2}\right)t} + e^{\left(\frac{-3-\sqrt{5}}{2}\right)t}}$

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

If $k^2 < 4\omega^2$, then there will

If $k^2 = 4\omega^2$, then there will

If $k^2 > 4\omega$, then there will

1. A particle P of mass 0.5kg moves in a horizontal straight line. At time t seconds, the displacement of P from a fixed point O, on the line is $x\ m$ and the velocity of P is $v\ ms^{-1}$. A force of magnitude $8x\ N$ acts on P in the direction PO. The particle is also subject to a resistance of magnitude $4v\ N$. When $t = 0$, $x = 1.5$ and P is moving in the direction of x increasing with speed $4\ ms^{-1}$.

a) Show that $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

b) Find the value of x when $t = 1$

2. A particle P hangs freely in equilibrium attached to one end of a light elastic string. The other end of the string is attached to a fixed point A. The particle is pulled down and held at rest in a container of liquid which exerts a resistance on the motion on P. P is then released from rest. While the string remains taut and the particle in the liquid, the motion can be modelled using the equation:

$$\frac{d^2x}{dt^2} + 6k\frac{dx}{dt} + 5k^2x = 0$$

Where k is a positive real constant.

Find the general solution to the differential equation and state the type of damping the particle is subject to.

3. One end of a light elastic spring is attached to a fixed point A. A particle P is attached to the other end and hangs in equilibrium vertically below A. The particle is pulled vertically down from its equilibrium position and released from rest. A resistance proportional to the speed of P acts on P.

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$$

The equation of motion of P is given as:
Where k is a positive real constant and x is the displacement of P from its equilibrium position.

a) Find the general solution to the differential equation.

b) Find the period of the motion

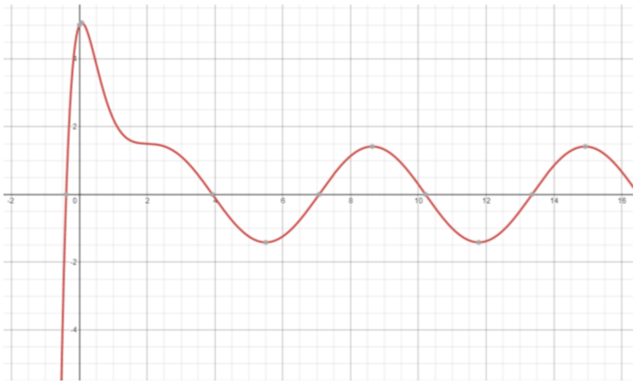
8C Part 2 Forced Harmonic Motion

1. A particle P of mass 1.5kg is moving along the x-axis. At time t the displacement of P from the origin O is x metres and the speed of P is $v\ ms^{-1}$. Three forces act on P, namely a restoring force of $7.5x\ N$, a resistance to motion of P of magnitude $6v\ N$ and a force of magnitude $12sint\ N$ acting in the direction OP. When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 2$.

a) Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 8sint$

b) Find x as a function of t

c) Describe the motion when t is large



$$x = e^{-2t}(6cost + 13sint) + sint - cost$$

2. A particle P is attached to end A of a light elastic string AB. Initially the particle and the string lie at rest on a smooth horizontal plane. At time $t = 0$, the end B of the string is set into motion and moves with constant speed U in the direction AB, and the extension in the string is x. Air resistance acting on P is proportional to its speed. The subsequent motion can be modelled by the differential equation:

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + k^2x = 2kU$$

Find an expression for x in terms of U, k and t.

8D Coupled Differential Equations

$$\frac{dx}{dt} = ax + by + f(t)$$

$$\frac{dy}{dt} = cx + dy + g(t)$$

1. At the start of 2010, a survey began on the number of bears and fish on a remote island in Northern Canada. After t years the number of bears, x, and the number of fish, y, in the area are modelled by the differential equations:

$$\frac{dx}{dt} = 0.3x + 0.1y$$

$$\frac{dy}{dt} = -0.1x + 0.5y$$

a) Show that $\frac{d^2x}{dt^2} - 0.8\frac{dx}{dt} + 0.16x = 0$

b) Find the general solution for the number of bears on the island at time t

c) Find the general solution for the number of fish on the island at time t

d) At the start of 2010, there were 5 bears and 20 fish on the island. Use this information to predict the number of bears on the island in 2020.

e) Comment on the suitability of the model

no other factors were taken into account as well – the f(t) and g(t) mentioned at the start

2. Two barrels contain contaminated water. At time t seconds, the amount of contaminant in barrel A is x ml and the amount in barrel B is y ml. Additional contaminated water flows into barrel A at a rate of 5 ml per second. Contaminated water flows from barrel A to barrel B and then back to barrel A through two connecting hoses, and then drains out of barrel A to leave the system completely. The system is modelled using the following differential equations:

$$\frac{dx}{dt} = 5 + \frac{4}{9}y - \frac{1}{7}x$$

$$\frac{dy}{dt} = \frac{3}{70}x - \frac{4}{9}y$$

Show that $630\frac{d^2y}{dt^2} + 370\frac{dy}{dt} + 28y = 135$