8) Parametric equations

8.1) Parametric equations

Chapter CONTENTS

Worked example

A curve has parametric equations

$$x = 3t, \qquad y = t^2, \qquad -4 < t < 4$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The domain and range of $f(x)$

c) Sketch the curve

Your turn

A curve has parametric equations

$$x = 2t, \qquad y = t^2, \qquad -3 < t < 3$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The domain and range of $f(x)$

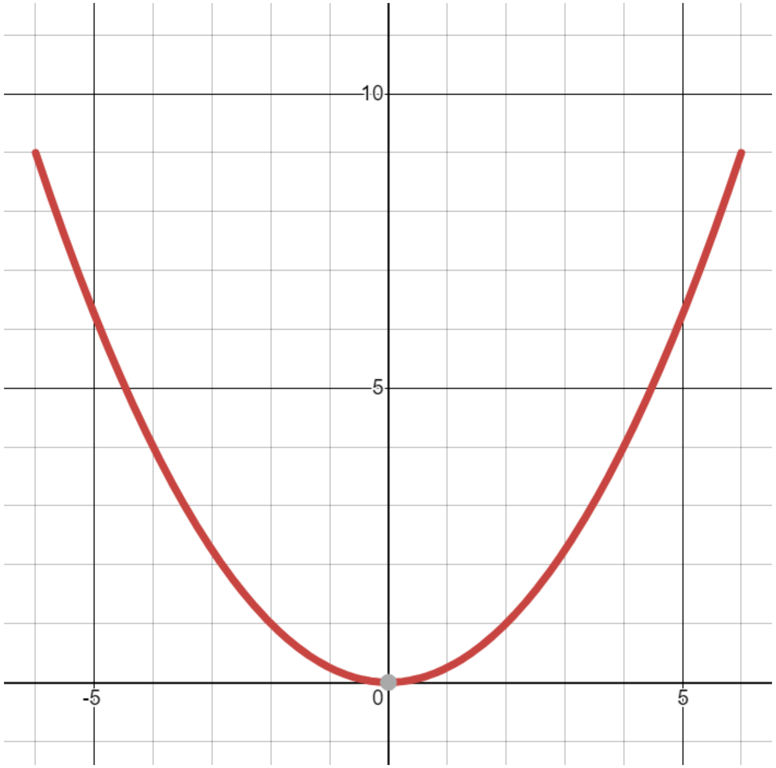
c) Sketch the curve

a) $y = \frac{x^2}{4}$

b) Domain: $-6 < x < 6$

Range: $0 \leq f(x) < 9$

c)



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Worked example

A curve has parametric equations

$$x = \ln(t + 5), \qquad y = \frac{1}{t + 7}, \qquad t > -4$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The domain and range of $f(x)$

Your turn

A curve has parametric equations

$$x = \ln(t + 3), \qquad y = \frac{1}{t + 5}, \qquad t > -2$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The domain and range of $f(x)$

a) $y = \frac{1}{e^x + 2}$

b) Domain: $x > 0$

Range: $0 < f(x) < \frac{1}{3}$

Worked example	Your turn
A curve has parametric equations $x = \ln t, \quad y = t^3 - 4, \quad t > 0$ Find: a) A Cartesian equation of the curve in the form $y = f(x)$ b) The domain and range of $f(x)$	A curve has parametric equations $x = \ln t, \quad y = t^2 - 1, \quad t > 0$ Find: a) A Cartesian equation of the curve in the form $y = f(x)$ b) The domain and range of $f(x)$ a) $y = e^{2x} - 1$ b) Domain: $x \in \mathbb{R}$ Range: $f(x) > -1$

Worked example

A curve has parametric equations

$$x = \frac{3t}{1-t}, \qquad y = 5t + \frac{2}{t},$$

Show that the Cartesian equation of the curve is

$$y = \frac{ax^2 + bx + c}{x(x+3)}$$

where a, b and c are constants to be found.

Your turn

A curve has parametric equations

$$x = \frac{5t}{1-t}, \qquad y = 2t + \frac{3}{t},$$

Show that the Cartesian equation of the curve is

$$y = \frac{ax^2 + bx + c}{x(x+5)}$$

where a, b and c are constants to be found.

Shown

$a = 5, b = 30, c = 75$

8.2) Using trigonometric identities

Chapter CONTENTS

Worked example

A curve has parametric equations

$$x = \sin t - 2, y = \cos t + 3, \quad t \in \mathbb{R}$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) Sketch the curve

Your turn

A curve has parametric equations

$$x = \sin t + 2, y = \cos t - 3, \quad t \in \mathbb{R}$$

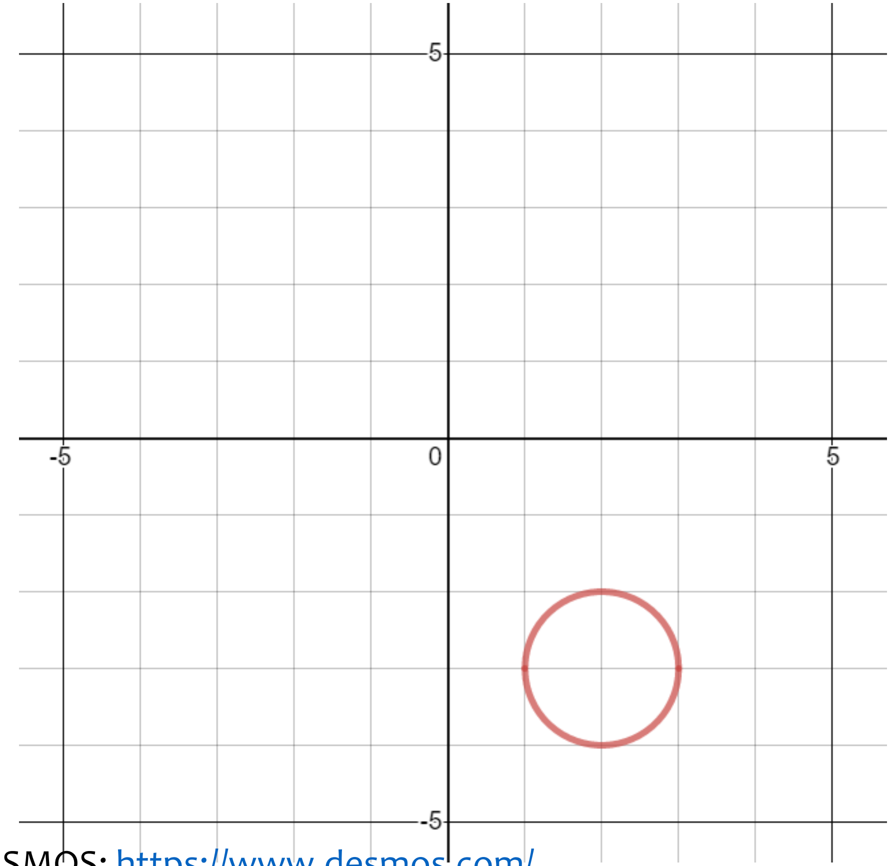
Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) Sketch the curve

a) $(x-2)^2 + (y+3)^2 = 1$

b) Circle, radius 1, centre $(2, -3)$



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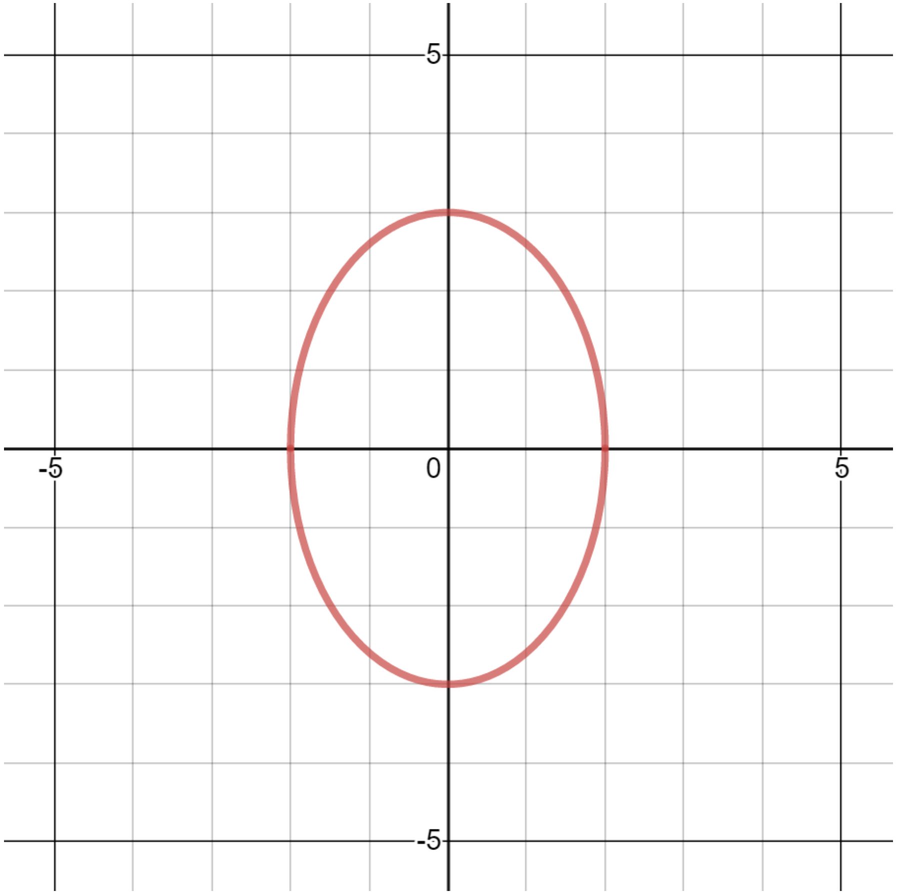
Worked example	Your turn
A curve has parametric equations $x = 2\sin t\,, y = 3\cos t\,,\quad t \in \mathbb{R}$ Find a Cartesian equation of the curve in the form $y = f(x)$	A curve has parametric equations $x = 3\sin t\,, y = 2\cos t\,,\quad t \in \mathbb{R}$ Find a Cartesian equation of the curve in the form $y = f(x)$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$



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Worked example	Your turn
A curve has parametric equations $$x = \cos t, \qquad y = \sin 2t, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$ Find: a) A Cartesian equation of the curve in the form $y = f(x)$ b) The valid domain and range of $f(x)$	A curve has parametric equations $$x = \sin t, \qquad y = \sin 2t, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$ Find: a) A Cartesian equation of the curve in the form $y = f(x)$ b) The valid domain and range of $f(x)$ a) $y = 2x\sqrt{1 - x^2}$ b) Domain: $-1 \le x \le 1$ Range: $-1 \le f(x) \le 1$

Worked example	Your turn
A curve has parametric equations $x = 4\cos t\,,\qquad y = \cos 2t - 1,\qquad 0 \le t \le \pi$ Find a Cartesian equation of the curve in the form $y = f(x), -k \le x \le k$, stating the value of the constant k	A curve has parametric equations $x = 2\sin t\,,\qquad y = 1 - \cos 2t\,,\qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$ Find a Cartesian equation of the curve in the form $y = f(x), -k \le x \le k$, stating the value of the constant k $y = \frac{x^2}{2}, -2 \le x \le 2\ (k = 2)$

Worked example	Your turn
A curve has parametric equations $x = \cot t + 1, \quad y = cosec^2\, t - 3, \quad 0 < t < \pi$ Find a Cartesian equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined	A curve has parametric equations $x = \cot t + 2, \quad y = cosec^2\, t - 2, \quad 0 < t < \pi$ Find a Cartesian equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined $y = x^2 - 4x + 3, x \in \mathbb{R}$

Worked example

A curve has parametric equations

$$x = \sqrt{5}\sin 2t\,, \qquad y = 10\sin^2 t\,, \qquad 0 \le t < \pi$$

Find a Cartesian equation of the curve

Your turn

A curve has parametric equations

$$x = \sqrt{3}\sin 2t\,, \qquad y = 4\cos^2 t\,, \qquad 0 \le t < \pi$$

Find a Cartesian equation of the curve

$$x^2 = 3y\left(1 - \frac{y}{4}\right)$$

Worked example	Your turn
A curve has parametric equations $x = 2\sin t, \quad y = \sin\left(t + \frac{\pi}{6}\right), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$ Find a Cartesian equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined	A curve has parametric equations $x = 2\cos t, \quad y = \sin\left(t - \frac{\pi}{6}\right), \quad 0 < t < \pi$ Find a Cartesian equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined $y = \frac{1}{4}\left(\sqrt{12 - 3x^2} - x\right), -2 < x < 2$

Worked example

A curve has parametric equations

$$x = \tan t\,, \qquad y = 5\sin(t - \pi)\,, \qquad 0 < t < \frac{\pi}{2}$$

Find a Cartesian equation of the curve

Your turn

A curve has parametric equations

$$x = \tan t\,, \qquad y = 4\sin(t + \pi)\,, \qquad 0 < t < \frac{\pi}{2}$$

Find a Cartesian equation of the curve

$$x = -\frac{y}{\sqrt{16 - y^2}}$$

8.3) Curve sketching

Worked example

Draw the curve given by the parametric equations

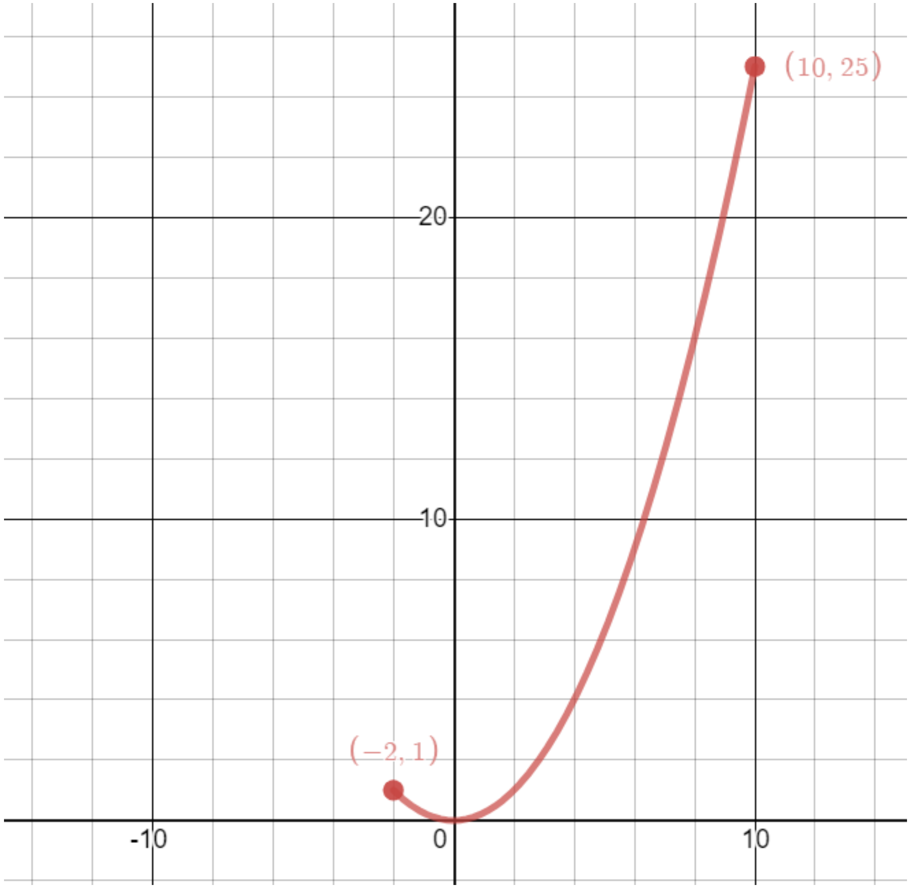
$$x = 3t, \qquad y = t^2, \qquad -5 \leq t \leq 1$$

Your turn

Draw the curve given by the parametric equations

$$x = 2t, \qquad y = t^2, \qquad -1 \leq t \leq 5$$

$$y = \frac{x^2}{4}, -2 \leq x \leq 10$$



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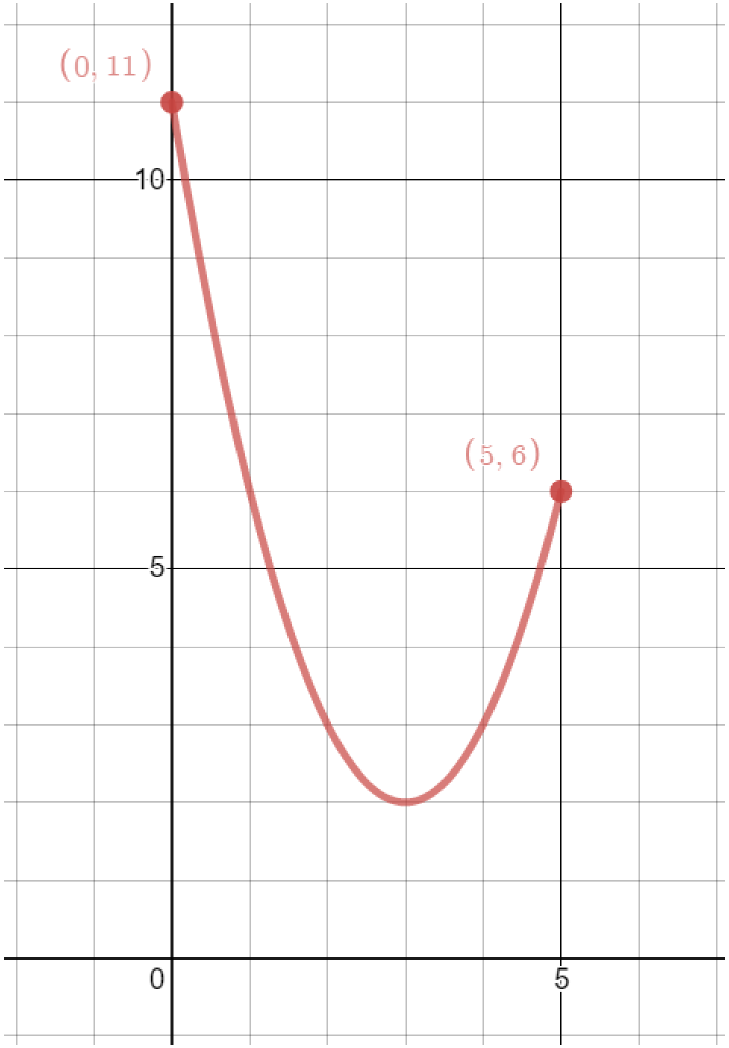
Worked example	Your turn
Draw the curve given by the parametric equations $x = 2 - t, \quad y = t^2 - 3, \quad -3 \leq t \leq 2$	Draw the curve given by the parametric equations $x = 3 - t, \quad y = t^2 + 2, \quad -2 \leq t \leq 3$

$y = x^2 - 6x + 11, 0 \leq x \leq 5$



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Worked example

Draw the curve given by the parametric equations

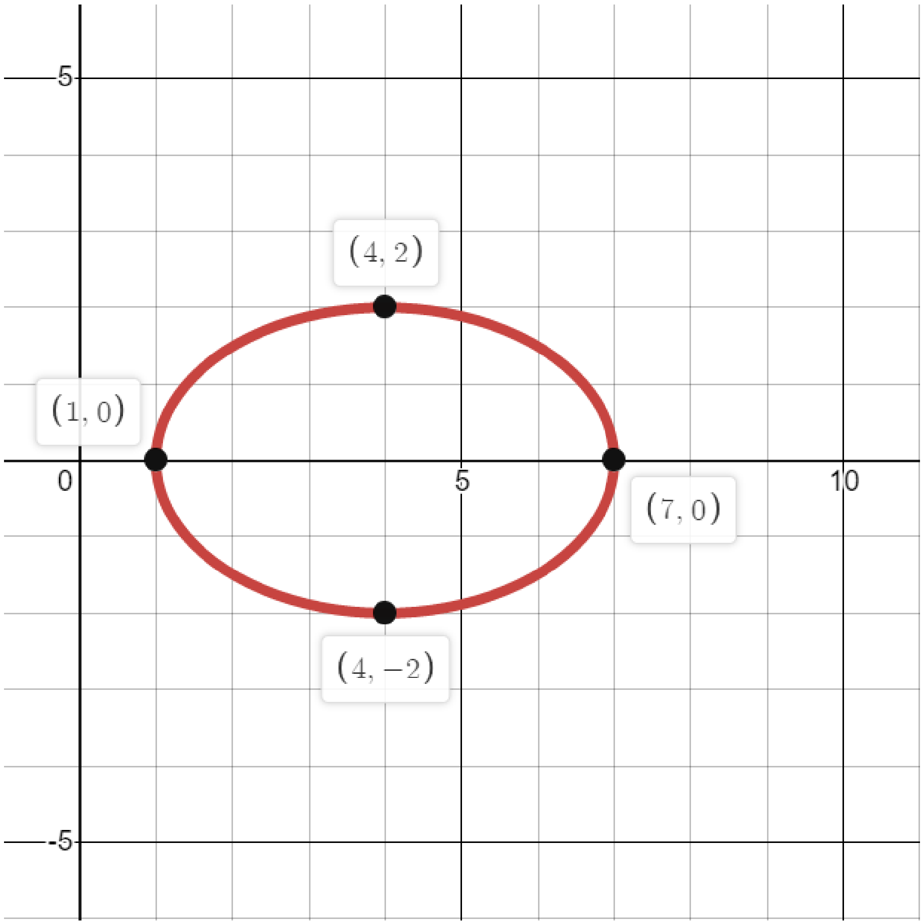
$x = 2\cos t - 3, \quad y = 4\sin t, \quad 0 \le t \le 2\pi$

Your turn

Draw the curve given by the parametric equations

$x = 3\cos t + 4, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

$$\left(\frac{x-4}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1, \quad 1 \le x \le 7$$



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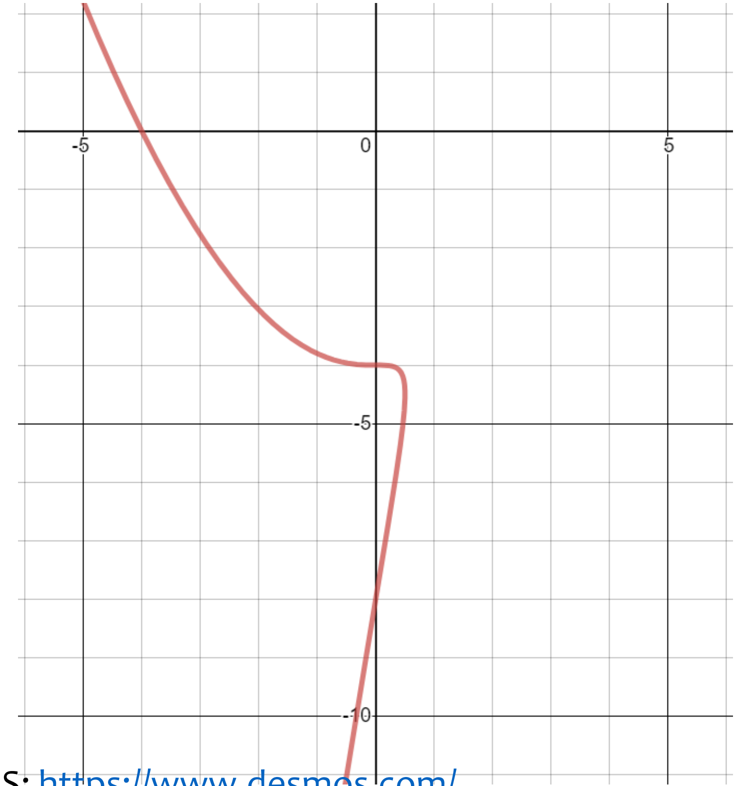
8.4) Points of intersection

Worked example	Your turn
A curve C is given by the parametric equations $x = at^2 + t, \quad y = a(t^3 + 27), \quad t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-6,0)$, a) find the value of a. b) find the coordinates of the points A and B where the curve crosses the y-axis.	A curve C is given by the parametric equations $x = at^2 + t, \quad y = a(t^3 + 8), \quad t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-4,0)$, a) find the value of a. b) find the coordinates of the points A and B where the curve crosses the y-axis.

a) $a = -\frac{1}{2}$

b) $(0, -4)$ and $(0, -8)$



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Worked example

A curve C is given by the parametric equations

$$x = t^2, \;\; y = 2t, \;\; t \in \mathbb{R}$$

Find the coordinates of the point(s) of intersection between the curve C and the line $x + y - 8 = 0$

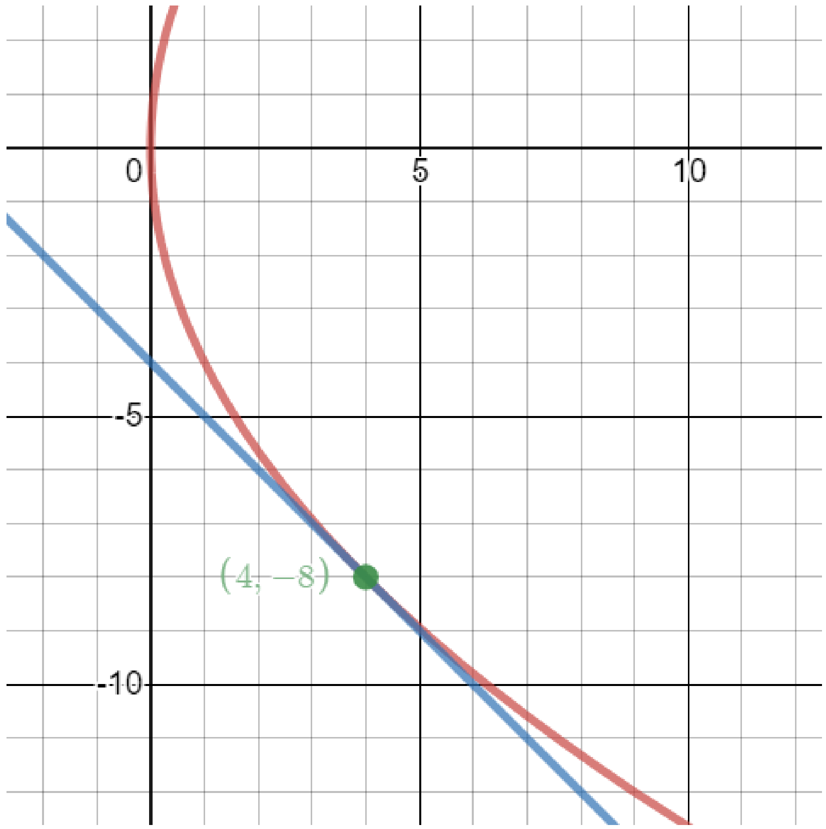
Your turn

A curve C is given by the parametric equations

$$x = t^2, \;\; y = 4t, \;\; t \in \mathbb{R}$$

Find the coordinates of the point(s) of intersection between the curve C and the line $x + y + 4 = 0$

$(4, -8)$



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Worked example	Your turn
A curve C is given by the parametric equations $x = \cos t - \sin t, \quad y = \left(t + \frac{\pi}{6}\right)^2, \quad -\frac{\pi}{3} < t < \frac{3\pi}{2}$ a) Find the point where the curve intersects the line $y = \pi^2$. b) Find the coordinates of the points where the curve cuts the y-axis.	A curve C is given by the parametric equations $x = \cos t + \sin t, \quad y = \left(t - \frac{\pi}{6}\right)^2, \quad -\frac{\pi}{2} < t < \frac{4\pi}{3}$ a) Find the point where the curve intersects the line $y = \pi^2$. b) Find the coordinates of the points where the curve cuts the y-axis. a) $\left(-\frac{1+\sqrt{3}}{2}, \pi^2\right)$ b) $\left(0, \frac{25\pi^2}{144}\right)$ and $\left(0, \frac{49\pi^2}{144}\right)$

Worked example

A curve C is given by the parametric equations

$$x = 1 - \frac{1}{3}t, \qquad y = 3^t - 1, \qquad t \in \mathbb{R}$$

Find the coordinates of the x and y intercepts

Your turn

A curve C is given by the parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1, \qquad t \in \mathbb{R}$$

Find the coordinates of the x and y intercepts

$(0, 3)$ and $(1, 0)$

Worked example	Your turn
A curve C is given by the parametric equations $$x = e^{3t}, \qquad y = e^{t} + 1, \qquad t \in \mathbb{R}$$ A straight line l passes through the points A and B where $t = \ln 3$ and $t = \ln 4$ respectively. Find an equation for l in the form $ax + by + c = 0$	A curve C is given by the parametric equations $$x = e^{2t}, \qquad y = e^{t} - 1, \qquad t \in \mathbb{R}$$ A straight line l passes through the points A and B where $t = \ln 2$ and $t = \ln 3$ respectively. Find an equation for l in the form $ax + by + c = 0$ $$x - 5y + 1 = 0$$

8.5) Modelling with parametric equations

Worked example

A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t \text{ m}, \qquad y = (v\sin\theta)t \text{ m}, \qquad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.

When the plane has travelled 500m horizontally, it has climbed 125m.

Given that the plane's speed is 40 m s^{-1}

a) find the parametric equations for the plane's motion.

b) find the vertical height of the plane after 20 seconds.

c) show that the plane's motion is a straight line.

d) explain why the domain of t, $t > 0$, is not realistic.

Your turn

A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t \text{ m}, \qquad y = (v\sin\theta)t \text{ m}, \qquad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.

When the plane has travelled 600m horizontally, it has climbed 120m.

Given that the plane's speed is 50 m s^{-1},

a) find the parametric equations for the plane's motion.

b) find the vertical height of the plane after 10 seconds.

c) show that the plane's motion is a straight line.

a) $x = 49.0t, y = 9.80t$ (3 sf)

b) $98\ m$

c) $y = \frac{1}{5}x$ which is linear

Worked example	Your turn
The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations	The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

Worked example

The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 4\cos 10t\,, \qquad y = 6\sin\left(5t - \frac{\pi}{3}\right), \qquad t \geq 0$$

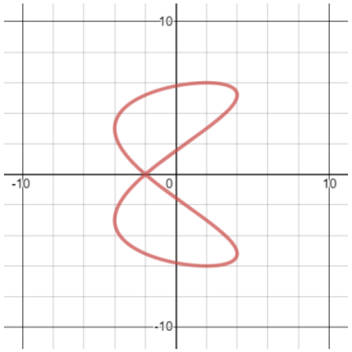
where x and y are measured in metres.

a) Find the coordinates of the figure skater at the beginning of his motion.

b) Find the coordinates of the point where the figure skater intersects his own path.

c) Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d) Determine how long it takes the figure skater to complete one complete figure-of-eight motion.



Your turn

The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 8\cos 20t\,, \qquad y = 12\sin\left(10t - \frac{\pi}{3}\right), \qquad t \geq 0$$

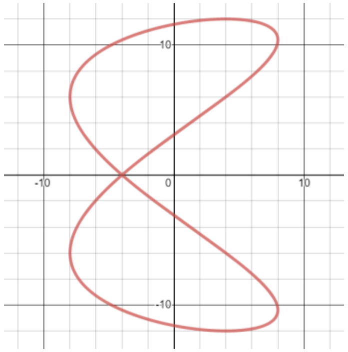
where x and y are measured in metres.

a) Find the coordinates of the figure skater at the beginning of his motion.

b) Find the coordinates of the point where the figure skater intersects his own path.

c) Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d) Determine how long it takes the figure skater to complete one complete figure-of-eight motion.



a) $t = 0, x = 8, y = -6\sqrt{3}$

b) $(-4, 0)$

c) $(0, -3.11), (0, 11.59), (0, 3.11), (0, -11.59)$ (2 dp)

d) $\frac{\pi}{5}$ minutes $= 37.7$ seconds (1 dp)

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Worked example

A stone is thrown from the top of a 50 m high cliff with an initial speed of 5 ms^{-1} at an angle of 30° above the horizontal. Its position after t seconds can be described using the parametric equations

$$x = \frac{5\sqrt{3}}{2}t\ m, \qquad y = \left(-4.9t^2 + \frac{5\sqrt{3}}{2}t + 50\right)m, \qquad 0 \le t \le k$$

where x is the horizontal distance, y is the vertical distance from the ground and k is a constant.

Given that the model is valid from the time the stone is thrown to the time it hits the ground,

a) find the value of k
b) find the horizontal distance travelled by the stone once it hits the ground

Your turn

A stone is thrown from the top of a 25 m high cliff with an initial speed of 5 ms^{-1} at an angle of 45° above the horizontal. Its position after t seconds can be described using the parametric equations

$$x = \frac{5\sqrt{2}}{2}t\ m, \qquad y = \left(-4.9t^2 + \frac{5\sqrt{2}}{2}t + 25\right)m, \qquad 0 \le t \le k$$

where x is the horizontal distance, y is the vertical distance from the ground and k is a constant.

Given that the model is valid from the time the stone is thrown to the time it hits the ground,

a) find the value of k
b) find the horizontal distance travelled by the stone once it hits the ground

a) $k = 2.65$ (2 dp)

b) 9.36 m (2 dp)