7) Algebraic methods

7.1) Algebraic fractions

Chapter CONTENTS

Worked example	Your turn
Simplify:	Simplify:

Worked example:

$$\frac{x^3}{x}$$

$$\frac{x^4}{x}$$

$$\frac{x^4}{x^2}$$

$$\frac{6x^4}{x^2}$$

$$\frac{6x^5}{2x^2}$$

$$\frac{6x^5}{8x^2}$$

$$\frac{6x^5}{12x^2}$$

$$\frac{6x^5}{18x^8}$$

Your turn:

$$\frac{8x^7}{12x^3}$$

$$\frac{2x^4}{3}$$

$$\frac{9x^5}{15x^9}$$

$$\frac{3}{5x^4}$$

Worked example	Your turn
Simplify: $$\frac{x^2+x-6}{9-x^2}$$	Simplify: $$\frac{4-x^2}{x^2+2x-8}$$ $$-\frac{x+2}{x+4}$$

Worked example	Your turn
Simplify: $$\frac{3x^2 \times 5y^3 \times 4x \times 2y^5}{10xy}$$ $$\frac{2x^3 \times 3y^5 \times x^4 \times 5y^2}{6x^2y^3}$$	Simplify: $$\frac{2a^4 \times 7b^3 \times 4a^2 \times 3b^7}{6a^3b^9}$$ $$28a^3b$$

Worked example	Your turn
Simplify: $$\frac{5x+10}{x+2}$$ $$\frac{x-7}{5x-35}$$	Simplify: $$\frac{x-3}{4x-12}$$ $$\frac{1}{4}$$

Worked example	Your turn
Simplify: $$\frac{x^2+5x+6}{x+2}$$ $$\frac{4x-12}{x^2-5x+6}$$	Simplify: $$\frac{x^2-7x+10}{3x-15}$$ $$\frac{x-2}{3}$$

Worked example	Your turn
Simplify:	Simplify:
$$\frac{x+3}{(x+3)^3}$$	$$\frac{(x-5)^2}{(x-5)^6}$$ $$\frac{1}{(x-5)^4}$$
$$\frac{(x-4)^5}{(x-4)^2}$$	$$\frac{(x+5)^3}{3x+15}$$ $$\frac{(x+5)^2}{3}$$
$$\frac{2x+10}{(x+5)^7}$$	

Worked example	Your turn
Simplify: $$\frac{3x^2 + 6x}{15x}$$ $$\frac{5y^2 - 30y^4}{10y^3}$$	Simplify: $$\frac{4x^3 + 12x}{16x^2}$$ $$\frac{x^2 + 3}{4x}$$

Worked example	Your turn
Simplify: $$\frac{x^2+6x+5}{x^2+3x-10}$$ $$\frac{x^2-7x-8}{x^2-1}$$	Simplify: $$\frac{x^2-6x+5}{x^2+4x-5}$$ $$\frac{x-5}{x+5}$$

Worked example	Your turn
Simplify: $$\frac{2x^2 - 5x - 3}{3x^2 - 11x + 6}$$ $$\frac{3x^2 - x - 10}{x^2 - 4}$$	Simplify: $$\frac{3x^2 - 5x - 2}{2x^2 - 7x + 6}$$ $$\frac{3x + 1}{2x - 3}$$

Worked example	Your turn
Simplify: $$\frac{x^2-1}{x+1}$$ $$\frac{x-2}{x^2-4}$$ $$\frac{x^2-9}{2x-6}$$	Simplify: $$\frac{2x+8}{x^2-16}$$ $$\frac{2}{x-4}$$

Worked example	Your turn
Simplify: $$\frac{x+2}{x-3} \times \frac{x-3}{x-2}$$ $$\frac{2x+3}{x-4} \times \frac{2x-8}{2x+1}$$	Simplify: $$\frac{3x-4}{x+2} \times \frac{5x+10}{3x+4}$$ $$\frac{5(3x-4)}{3x+4}$$

Worked example	Your turn
Simplify: $$\frac{x+2}{x-3} \div \frac{x-2}{x-3}$$ $$\frac{2x+3}{x-4} \div \frac{2x+1}{2x-8}$$	Simplify: $$\frac{3x-4}{x+2} \div \frac{3x+4}{5x+10}$$ $$\frac{5(3x-4)}{3x+4}$$

Worked example	Your turn
Simplify: $$\frac{(x+2)(3x-4)}{(x-5)(6x+7)} \times \frac{(7x+6)(x-5)}{(4x-3)(x+2)}$$	Simplify: $$\frac{(x+5)(2x-7)}{(x-4)(3x+1)} \times \frac{(6x+1)(x-4)}{(2x-9)(x+3)}$$ $$\frac{(x+5)(2x-7)(6x+1)}{(3x+1)(2x-9)(x+3)}$$
$$\frac{(x+7)(6x-5)}{(x-4)(3x+2)} \times \frac{(x+4)(3x+2)}{(6x+5)(7x)}$$	

Worked example	Your turn
Simplify: $$\frac{3x^2-10x-8}{6x^2+37x-35} \div \frac{x^2-3x-4}{x^2-49}$$	Simplify: $$\frac{2x^2-7x-15}{3x^2+10x-8} \div \frac{2x^2+x-3}{x^2-16}$$ $$\frac{(x-5)(x-4)}{(3x-2)(x-1)}$$

Worked example	Your turn
Simplify: $$\frac{2x^3 - 5x^2 - 3x}{2x - 6}$$	Simplify: $$\frac{3x^3 - x^2 - 10x}{4x - 8}$$ $$\frac{x(3x + 5)}{4}$$

Worked example	Your turn
Simplify: $$\frac{2x^3 + 5x^2 - 3x}{4x^2 - 1}$$	Simplify: $$\frac{3x^3 - x^2 - 10x}{9x^2 - 25}$$ $$\frac{x(x-2)}{3x-5}$$

Worked example	Your turn
Write as a single fraction:	Write as a single fraction:
$$3+\frac{5}{2x-1}$$	$$5-\frac{3}{x+2}$$
	$$\frac{5x+7}{x+2}$$
$$2-\frac{3}{5x+4}$$	

Worked example	Your turn
Write as a single fraction: $$3-(x-4)\div\frac{x^2-16}{x-5}$$	Write as a single simplified fraction: $$5-(x-2)\div\frac{x^2-4}{x+3}$$ $$\frac{4x-13}{x-2}$$

Worked example	Your turn
Write in the form $1+\frac{a}{x+b}$:	Write in the form $1+\frac{a}{x+b}$:
$$\frac{x+3}{x-5}$$	$$\frac{x-5}{x+2}$$
	$$1-\frac{7}{x+2}$$
$$\frac{x-2}{x+7}$$	

Worked example	Your turn
Simplify: $$\frac{\frac{3}{x}+2}{1+\frac{5}{x}}$$ $$\frac{\frac{3}{2x}-2}{1-\frac{5}{2x}}$$	Simplify: $$\frac{\frac{5}{3x}+3}{7+\frac{2}{3x}}$$ $$\frac{9x+5}{21x+2}$$

7.2) Dividing polynomials

Worked example	Your turn
Divide $x^3 - 2x^2 - 17x + 10$ by $(x - 5)$	Divide $x^3 + 2x^2 - 17x + 6$ by $(x - 3)$ $x^2 + 5x - 2$

Worked example	Your turn
$f(x) = 18x^4 - 29x^2 + 3$ Divide $f(x)$ by $(3x+1)$. Give your answer in the form $f(x) = (3x+1)(ax^3 + bx^2 + cx + d)$	$f(x) = 4x^4 - 17x^2 + 4$ Divide $f(x)$ by $(2x+1)$. Give your answer in the form $f(x) = (2x+1)(ax^3 + bx^2 + cx + d)$ $f(x) = (2x+1)(2x^3 - x^2 - 8x + 4)$

Worked example	Your turn
Find the remainder when $2x^3 + 5x^2 - 10x + 16$ is divided by $(x - 2)$	Find the remainder when $2x^3 - 5x^2 - 16x + 10$ is divided by $(x - 4)$ -6

Worked example	Your turn
Divide $27x^3 - 8$ by $3x - 2$	Divide $8x^3 - 1$ by $2x - 1$ $4x^2 + 2x + 1$

Worked example	Your turn
$f(x) = 6x^3 + 11x^2 - 46x + 24$ Show that $(3x - 2)$ is a factor of $f(x)$ and hence find all the real roots of the equation $f(x) = 0$	$f(x) = 12x^3 - 14x^2 - 61x + 60$ Show that $(2x - 3)$ is a factor of $f(x)$ and hence find all the real roots of the equation $f(x) = 0$ $x = -\frac{5}{2}, x = \frac{3}{2}, x = \frac{4}{3}$

7.3) The factor theorem

Chapter CONTENTS

Worked example	Your turn
Show that $(x-3)$ is a factor of x^3-2x^2-5x+6	Show that $(x-2)$ is a factor of x^3+x^2-4x-4 Shown (e.g. algebraic division or factor theorem)

Worked example	Your turn
Fully factorise $3x^3 + x^2 - 12x - 4$	Fully factorise $2x^3 + x^2 - 18x - 9$ $(x-3)(2x+1)(x+3)$

Worked example	Your turn
Given that $x + 2$ is a factor of $3x^4 - 4x^2 + a$, find the value of a.	Given that $x + 1$ is a factor of $4x^4 - 3x^2 + a$, find the value of a. $a = -1$

Worked example	Your turn
Given that $3x + 1$ is a factor of $12x^3 + ax^2 + 2$, find the value of a.	Given that $2x + 1$ is a factor of $6x^3 + ax^2 + 1$, find the value of a. $a = -1$

Worked example	Your turn
Given that $2x - 1$ is a factor of $2x^3 + 3x^2 + ax + 11$, find the value of a.	Given that $3x - 1$ is a factor of $3x^3 + 11x^2 + ax + 1$, find the value of a. $a = -7$

Worked example	Your turn
Show that $(x-2)$ is a factor of $5x^4-16x^3-47x^2+130x-24$ and hence find all the real solutions to $5x^4-16x^3-47x^2+130x-24=0$	Show that $(x-3)$ is a factor of $4x^4+15x^3-48x^2-109x+30$ and hence find all the real solutions to $4x^4+15x^3-48x^2-109x+30=0$ $x=-5, x=-2, x=\frac{1}{3}, x=3$

7.4) Mathematical proof

Worked example	Your turn
Prove that $(2x-3)(x-7)(x+5) \equiv 2x^3 - 7x^2 - 64x + 105$	Prove that $(3x+2)(x-5)(x+7) \equiv 3x^3 + 8x^2 - 101x - 70$ Proof

Worked example	Your turn
Prove that if three consecutive even integers are the sides of a right-angled triangle, they must be 6, 8 and 10.	Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5. Proof e.g. Pythagoras' Theorem with $x, x+1, x+2$

Worked example	Your turn
Prove that $x^2 + 6x + 11$ is positive for all values of x.	Prove that $x^2 + 4x + 5$ is positive for all values of x.

$$x^2 + 4x + 5 = (x + 2)^2 + 1$$
$$k^2 \geq 0$$
$$(x + 2)^2 \geq 0$$
$$(x + 2)^2 + 1 \geq 1$$

Worked example	Your turn
Prove that the sum of the squares of two consecutive even numbers is 4 more than a multiple of 8.	Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8. Proof

Worked example	Your turn
Prove that $(1, 1)$, $(4,7)$ and $(10, 4)$ are the vertices of a right-angled triangle.	Prove that $(1, 1)$, $(3, 3)$ and $(4, 2)$ are the vertices of a right-angled triangle. Proof e.g. Pythagoras' Theorem or perpendicular gradients AB and BC

Worked example	Your turn
The equation $kx^2 + 5kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leq k < \frac{12}{25}$	The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leq k < \frac{8}{9}$ Proof

Worked example	Your turn
Prove that $4n - 3 + 2n + 7$ is a multiple of 2 for all real integers n	Prove that $4n - 3 + 10n - 11$ is a multiple of 7 for all real integers n
	$14n - 14$ $\equiv 7(2n - 2)$ $\equiv 7k$
Prove that $4n - 3 + 2n - 9$ is a multiple of 3 for all real integers n	

Worked example	Your turn
Prove that the sum of five consecutive integers is a multiple of 5.	Prove that the sum of three consecutive integers is a multiple of 3. Let the first integer be n: $n + n + 1 + n + 2$ $\equiv 3n + 3$ $\equiv 3(n + 1)$

Worked example	Your turn
Prove that the product of two odd numbers is an odd number.	Prove that the product of two even numbers is an even number.

Let even numbers be $2m$ and $2n$:

$$2m \times 2n$$
$$\equiv 4mn$$
$$\equiv 2(2mn)$$

Worked example	Your turn
Prove algebraically that $n^2 - 2 - (n-2)^2$ is always even, given n is an integer greater than 1	Prove algebraically that $(2n+1)^2 - (2n+1)$ is an even number $(2n+1)^2 - (2n+1)$ $\equiv 4n^2 + 4n + 1 - 2n - 1$ $\equiv 4n^2 + 2n$ $\equiv 2(2n^2 + n)$

Worked example	Your turn
Prove that $(n-1)^2 + n^2 + (n+1)^2$ is two more than a multiple of 3 for all positive integer values of n	Prove that $(n+1)^2 - n^2$ is one more than a multiple of 2 for all positive integer values of n $(n+1)^2 - n^2$ $\equiv n^2 + 2n + 1 - n^2$ $\equiv 2n + 1$

Worked example	Your turn
Prove that $(2n+3)^2-(2n-3)^2$ is a multiple of 8 for all positive integer values of n	Prove that $(3n+2)^2-(3n-2)^2$ is a multiple of 8 for all positive integer values of n $(3n+2)^2-(3n-2)^2$ $\equiv 9n^2+12n+4-(9n^2-12n+4)$ $\equiv 9n^2+12n+4-9n^2+12n-4$ $\equiv 24n$ $\equiv 8(3n)$

Worked example	Your turn
Prove algebraically that the difference between two different odd numbers is an even number.	Prove algebraically that the difference between two different even numbers is an even number. Let even numbers be $2m$ and $2n$: $2m - 2n$ $\equiv 2(m - n)$

Worked example	Your turn
Prove that the product of four consecutive integers is always a multiple of 8	Prove that the product of three consecutive integers is always a multiple of 6 Proof by showing at least one is a multiple of 2, and one will be a multiple of 3…

Worked example	Your turn
Prove that, for all positive values of n, $\frac{(n+3)^2-(n-2)^2}{2n^2+n} = \frac{a}{b}$ where a and b are integers or variables.	Prove that, for all positive values of n, $\frac{(n+2)^2-(n+1)^2}{2n^2+3n} = \frac{a}{b}$ and find the integers a and b $a = 1, b = n$

7.5) Methods of proof

Worked example	Your turn
Prove by exhaustion that the sum of an integer and the square of the integer is even for all integers n	Prove by exhaustion that all square numbers are either a multiple of 4 or 1 more than a multiple of 4 Proof

Worked example	Your turn
Disprove the statement: "$n^2 - n + 3$ is prime for all integers n."	Disprove the statement: "$n^2 - n + 41$ is prime for all integers n." $n = 41, n^2 - n + 41 = 41^2 = 1681$ 1681 has 3 factors: $1, 41$ and 1681

Worked example	Your turn
a) Prove that for all positive values of p and q $$p+q>\sqrt{4pq}$$ b) Use a counter-example to show that this is not true when p and q are not both positive.	a) Prove that for all positive values of x and y $$\frac{x}{y}+\frac{y}{x}\geq 2$$ b) Use a counter-example to show that this is not true when x and y are not both positive.

a) $(x-y)^2 \geq 0$

$$x^2+y^2-2xy \geq 0$$

$$\frac{x^2+y^2-2xy}{xy} \geq 0$$

[valid as $x, y > 0 \therefore xy > 0$]

$$\frac{x^2}{xy}+\frac{y^2}{xy}-\frac{2xy}{xy} \geq 0$$

$$\frac{x}{y}+\frac{y}{x}-2 \geq 0$$

$$\frac{x}{y}+\frac{y}{x} \geq 2$$

b) e.g. $x=-3, y=6$

$$-\frac{3}{6}+\frac{6}{-3}=-\frac{1}{2}-2=-\frac{5}{2}<-2$$