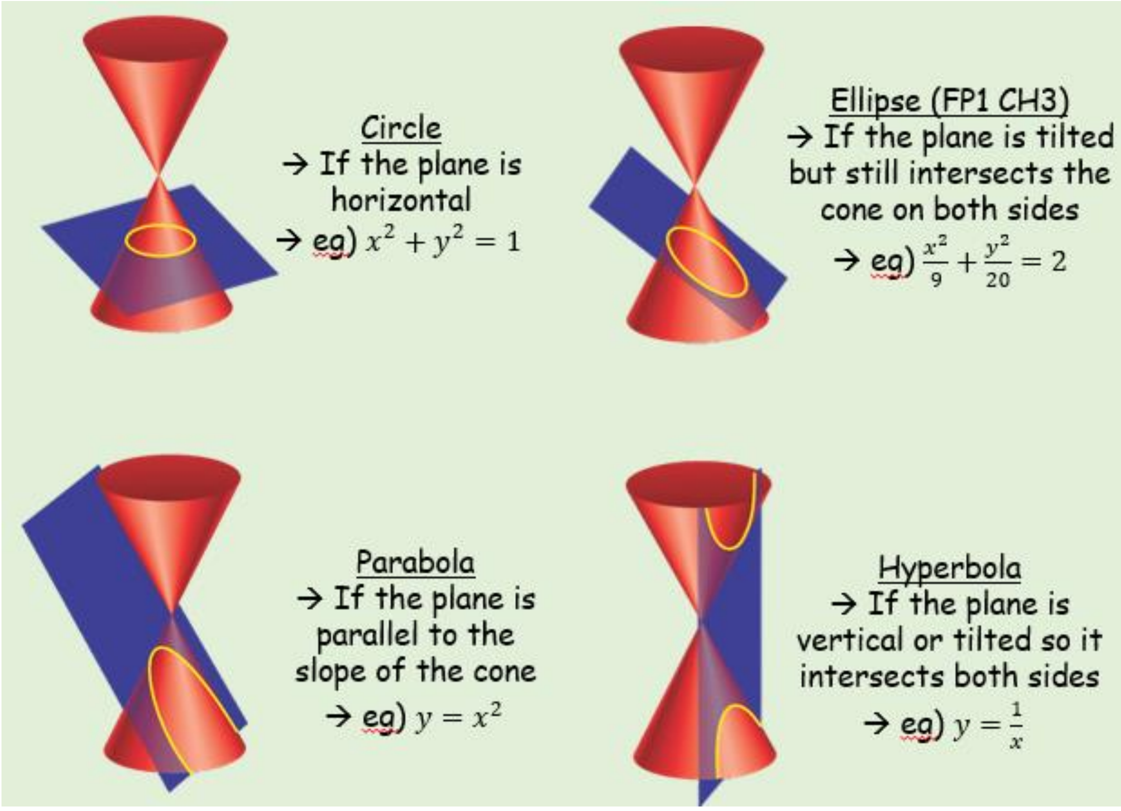
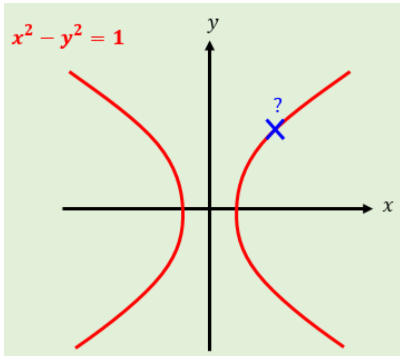
6A Introduction to Hyperbolic Functions





$sinhx$

$cosechx$

$coshx$

$sechx$

$tanhx$

$cothx$

1. Find, to 2 decimal places, the value of:

a) $\sinh 3$

b) $\cosh 1$

c) $\tanh 0.8$

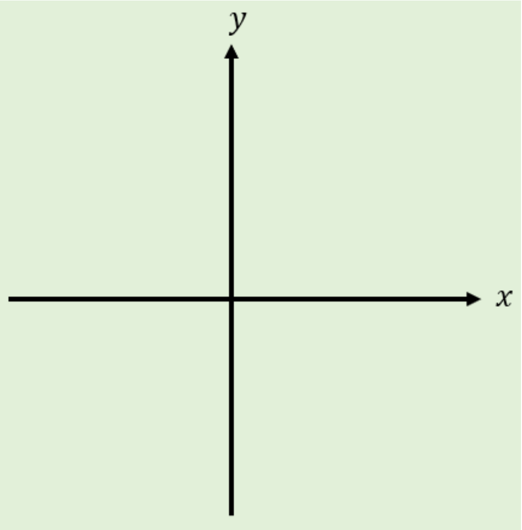
2. Find the exact value of:

$$\tanh(ln4)$$

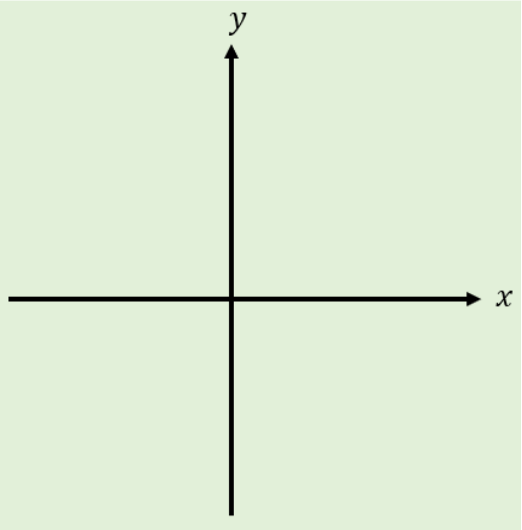
3. Find the value of x for which $\sinh x = 5$. Give your answer to 2 decimal places.

4. Sketch hyperbolic function

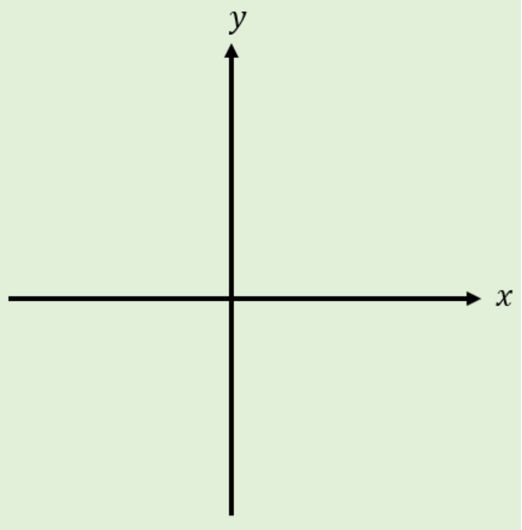
a) $y = \sinh x$



b) $\mathrm{y} = \cosh x$



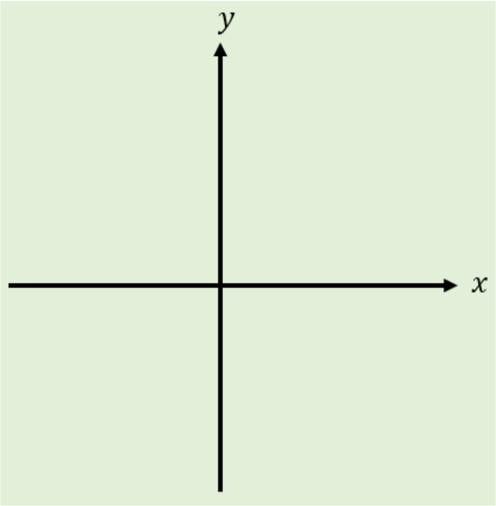
c) $y = tanh\, x$



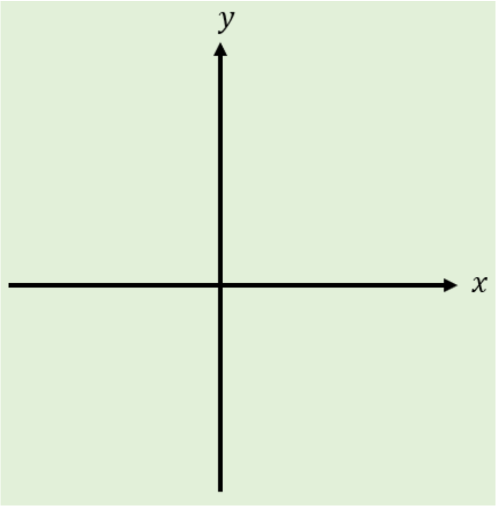
6B Inverse Hyperbolic Functions

1. Sketch hyperbolic function

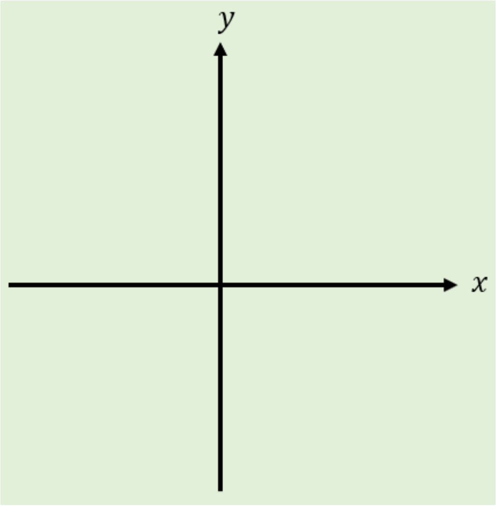
a) $y = arsinh\ x$



b) $\mathrm{y} = \operatorname{arcosh} x$



c) $y = artanh\, x$



2. Show that $arsinh\, x = ln\left(x + \sqrt{x^2 + 1}\right)$

3. Show that $arcosh\, x = ln\left(x + \sqrt{x^2 - 1}\right)$

4. Find an expression for $artanh\ x$ using logarithms

5. Express using natural logarithms:

a) $arsinh\ 1$

b) $arcosh\ 2$

c) $artanh\ \frac{1}{3}$

6C Hyperbolic Equations and Identities

1. Prove that:

a)

$$cosh^2 A - sinh^2 A \equiv 1$$

b)

$$\sinh(A + B) \equiv \sinh A \cosh B + \cosh A \sinh B$$

c)

$$\cosh 2A \equiv 1 + 2sinh^2A$$

Osborn’s Rule:

2. Write down the hyperbolic identity corresponding to:

a)

$$cos2A \equiv 2cos^2 A - 1$$

b)

$$\tan(A - B) \equiv \frac{tanA - tanB}{1 + tanAtanB}$$

3. Given that $\sinh x = \frac{3}{4}$, find the exact value of:

a) $\cosh x$

b) $\tanh x$

c) $\sinh 2x$

4. Solve the equation below for real values of x.

$$6\sinh x - 2\cosh x = 7$$

5. Solve the equation below, giving answers as natural logarithms.

$$2cosh^2x - 5\sinh x = 5$$

6. Solve the equation below, giving answers as natural logarithms where appropriate.

$$\cosh 2x - 5\cosh x + 4 = 0$$

Some additions to Osborn's rule

6D Differentiating Hyperbolics

1. Show that $\frac{d}{dx}(\sinh x) = \cosh x$

2. Show that $\frac{d}{dx}(\cosh x) = \sinh x$

3. Show that $\frac{d}{dx}(\tanh x) = sech^2\, x$

4. Differentiate $\cosh 3x$ with respect to x

5. Differentiate $x^2 \cosh 4x$ with respect to x

6. Given that:

$$y = A\cosh 3x + B\sinh 3x$$

Where A and B are constants, prove that $\frac{d^2y}{dx^2} = 9y$

7. Show that $\frac{d}{dx}(arcosh\,x) = \frac{1}{\sqrt{x^2-1}}$, $x > 1$

8. Given $y = xarcosh\,x$, find $\frac{dy}{dx}$

9. Given $y = (arcosh\, x)^2$, prove that:

$$(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y$$

10.

a) Show that $\frac{d}{dx}(arsinh\, x) = \frac{1}{\sqrt{1+x^2}}$

b) Find the first two non-zero terms in the series expansion of $arsinh\ x$

c) The general term for the series expansion of $arsinh\ x$ is given by:

$$arsinh\ x = \sum_{r=0}^{\infty} \left(\frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \right) \frac{x^{2n+1}}{2n+1}$$

Find, in its simplest terms, the third term in the sequence

d) Use your approximation, up to and including the term in x^5, to find an approximation for $arsinh\ 0.5$

e) Calculate the percentage error by using this approximation

6E Integrating Hyperbolics

1. Find

a)

$$\int \cosh(4x - 1)\ dx$$

b)

$$\int \left(\frac{2 + 5x}{\sqrt{x^2 + 1}}\right) dx$$

c)

$$\int cosh^5 2x\ sinh2x\ dx$$

d)

$$\int tanhx\ dx$$

e)

$$\int cosh^2 3x\ dx$$

f)

$$\int sinh^3 x \, dx$$

g)

$$\int e^{2x} \sinh x \;\; dx$$

2. By using an appropriate substitution, find:

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx \,, \; x > a$$

Formula book reference:

Hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2\sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\operatorname{arcosh} x = \ln\{x + \sqrt{x^2 - 1}\} \qquad (x \geqslant 1)$

$\operatorname{arsinh} x = \ln\{x + \sqrt{x^2 + 1}\}$

$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$

Integration (+ constant; $a > 0$ where relevant)

$f(x)$	$\int f(x)\,dx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x + \sqrt{x^2 + a^2}\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Differentiation

$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

3. Show that

a)

$$\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx = ln\left(\frac{2+\sqrt{3}}{2}\right)$$

b)

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}arsinhx + \frac{1}{2}x\sqrt{1+x^2} + c$$

4. By using a hyperbolic substitution, evaluate:

$$\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\,dx$$

5.

$$\int \frac{1}{\sqrt{12x + 2x^2}}\, dx$$

6. Use the substitution

$$x = \frac{1}{2}(3 + 4\cosh u)$$

to find:

$$\int \frac{1}{\sqrt{4x^2 - 12x - 7}}\, dx$$