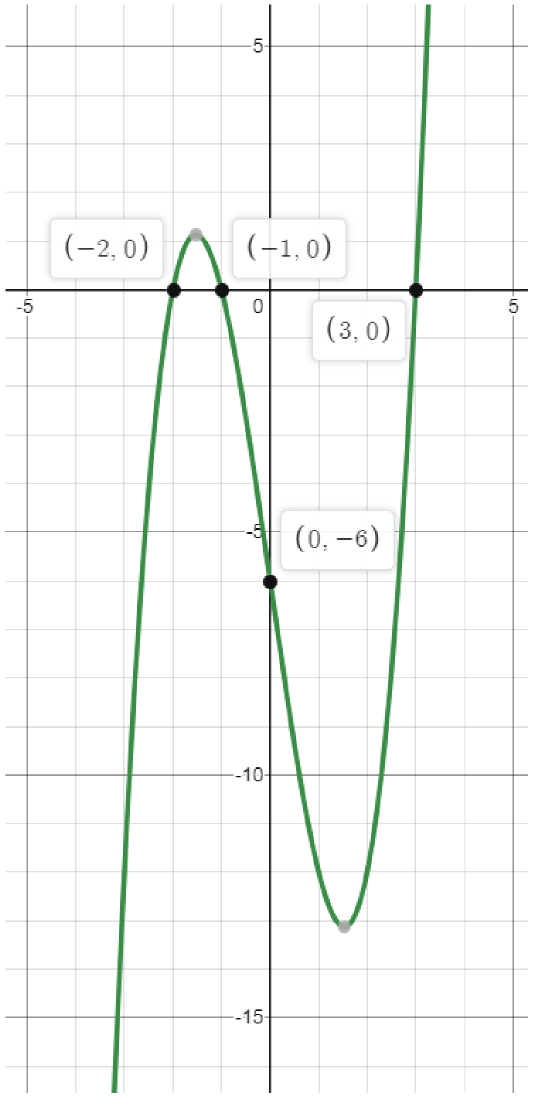
4) Graphs and transformations

4.1) Cubic graphs

Chapter CONTENTS

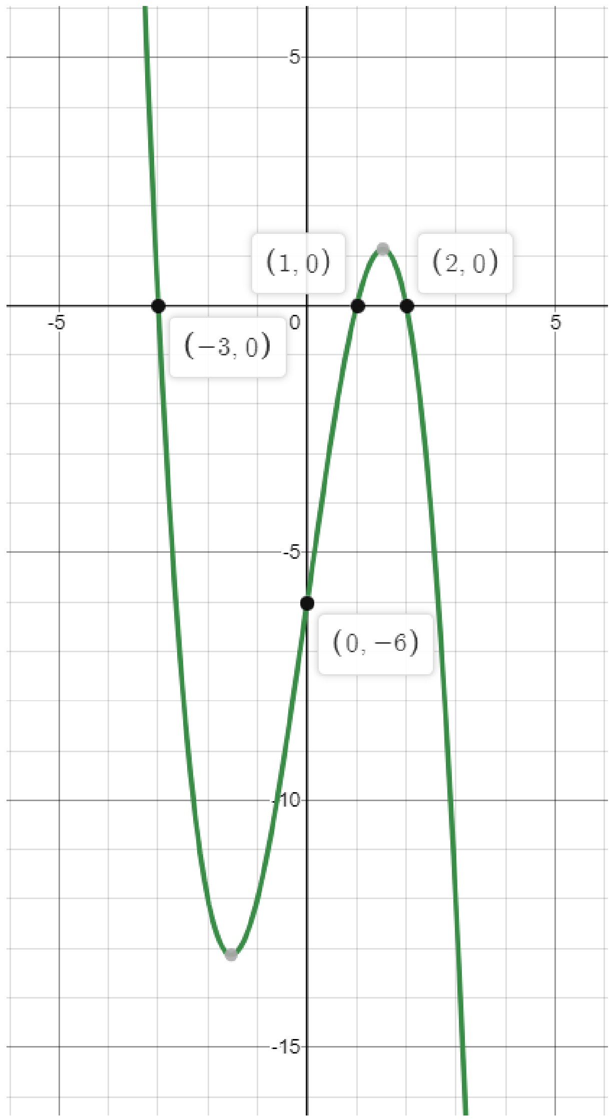
Worked example	Your turn
Sketch the graph of: $y = (x+1)(x+2)(x+3)$	Sketch the graph of: $y = (x+1)(x+2)(x-3)$

$$y = (x+1)(x-2)(x+3)$$

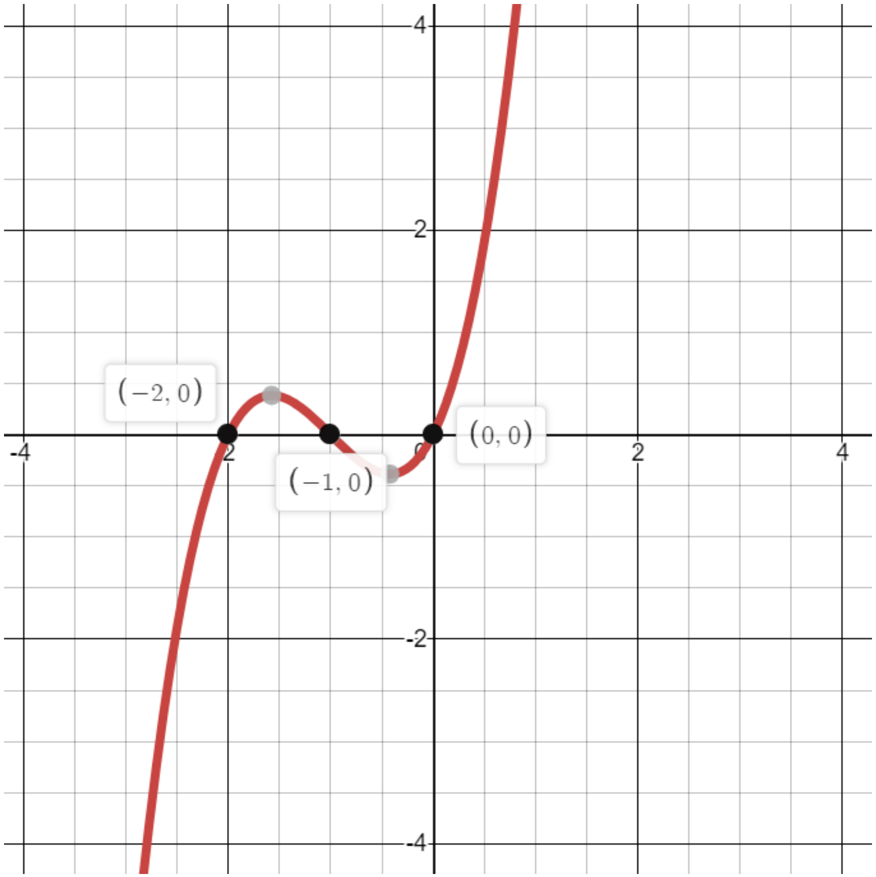


Worked example	Your turn
Sketch the graph of: $y=(x+1)(x-2)(3-x)$	Sketch the graph of: $y=(x-1)(x+3)(2-x)$

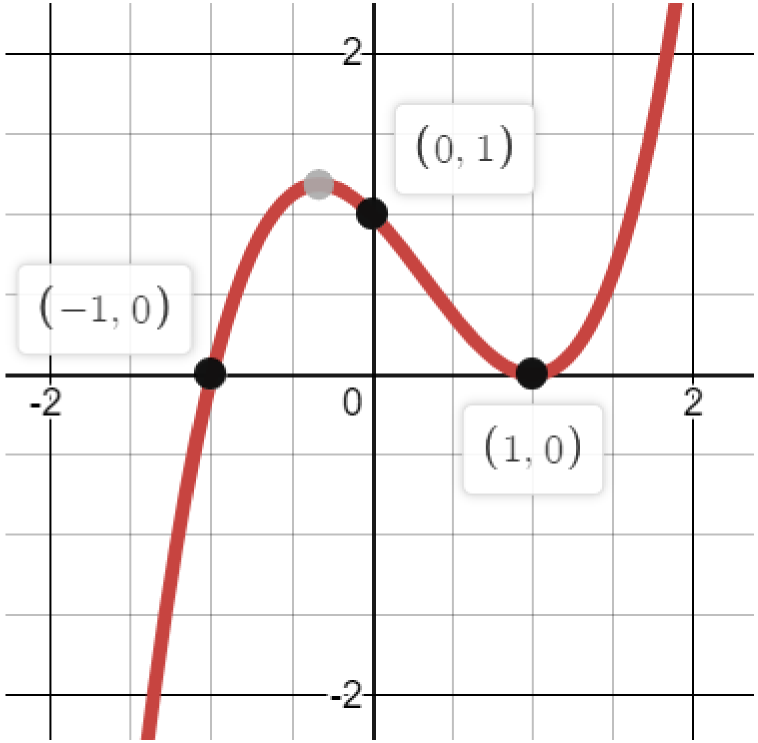
$$y=(x-1)(x-2)(3-x)$$



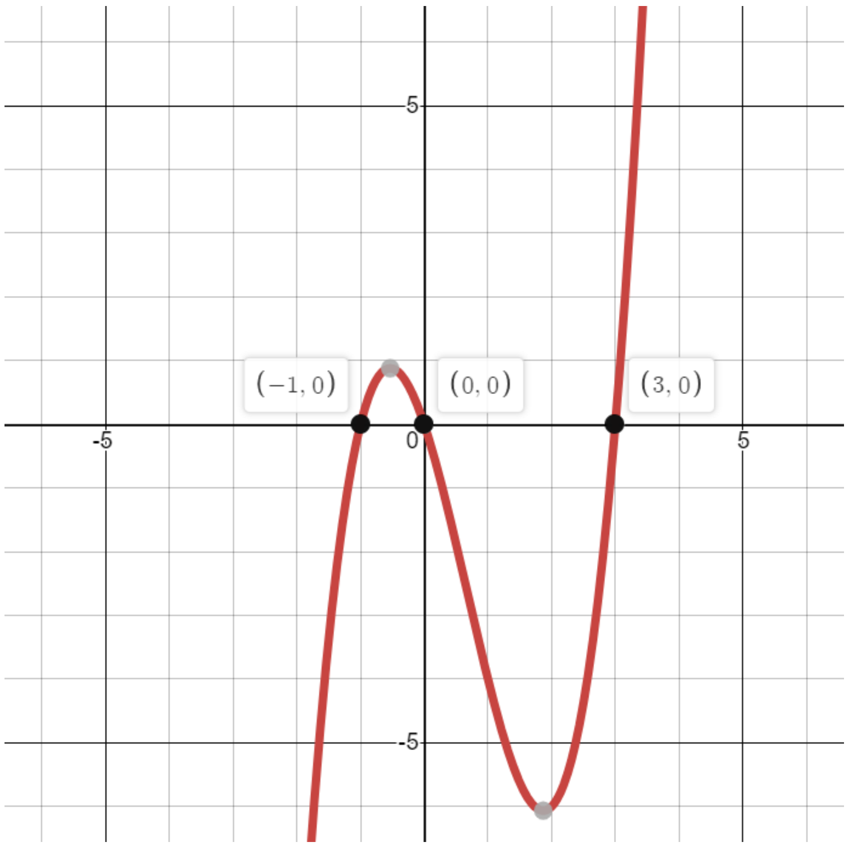
Worked example	Your turn
Sketch the graph of: $$y = x(x+3)(x+4)$$	Sketch the graph of: $$y = x(x+1)(x+2)$$



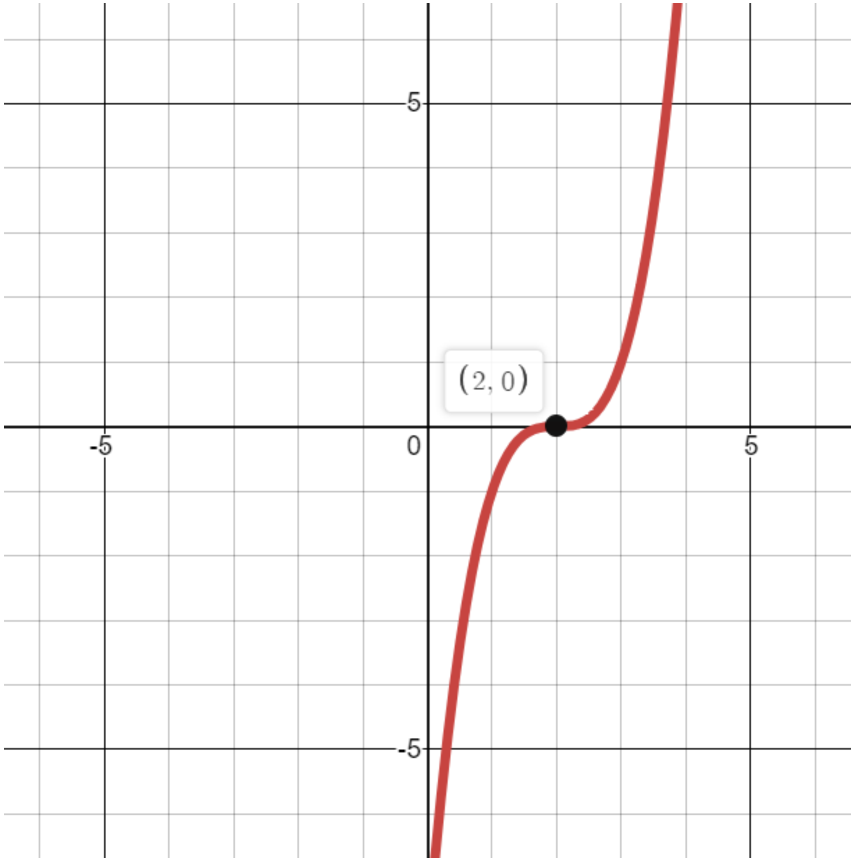
Worked example	Your turn
Sketch the graph of: $y = (x+2)^2(x-2)$	Sketch the graph of: $y = (x-1)^2(x+1)$



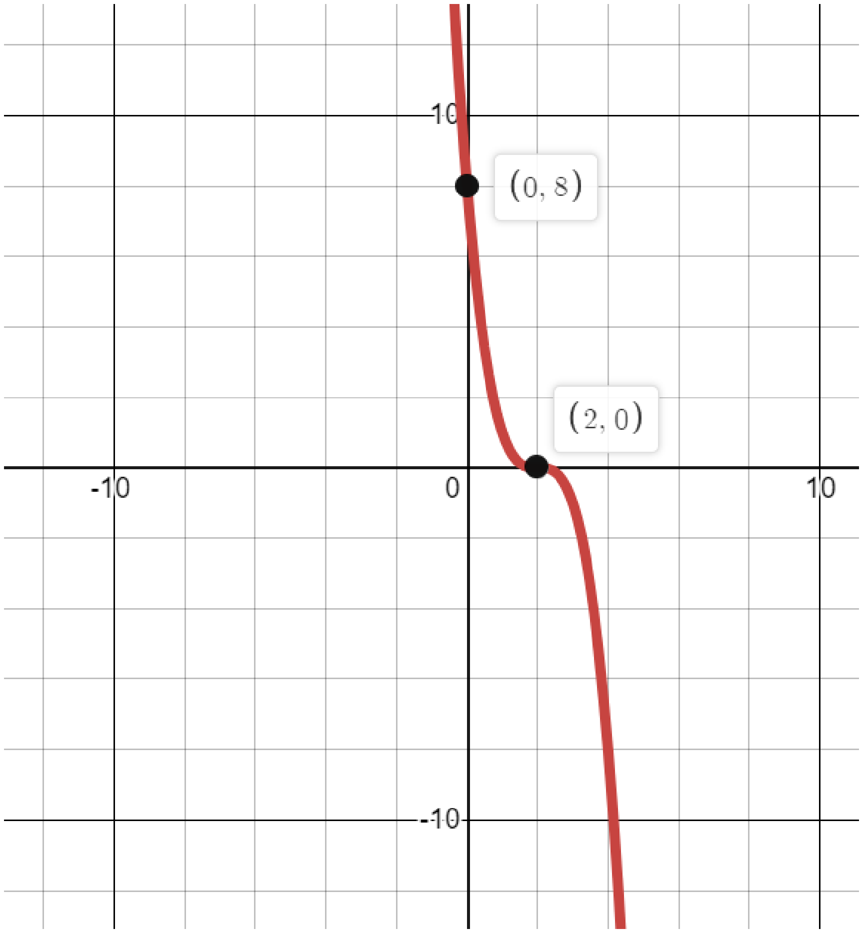
Worked example	Your turn
Sketch the graph of: $y = x^2 - 4x^2 - 5x$	Sketch the graph of: $y = x^3 - 2x^2 - 3x$



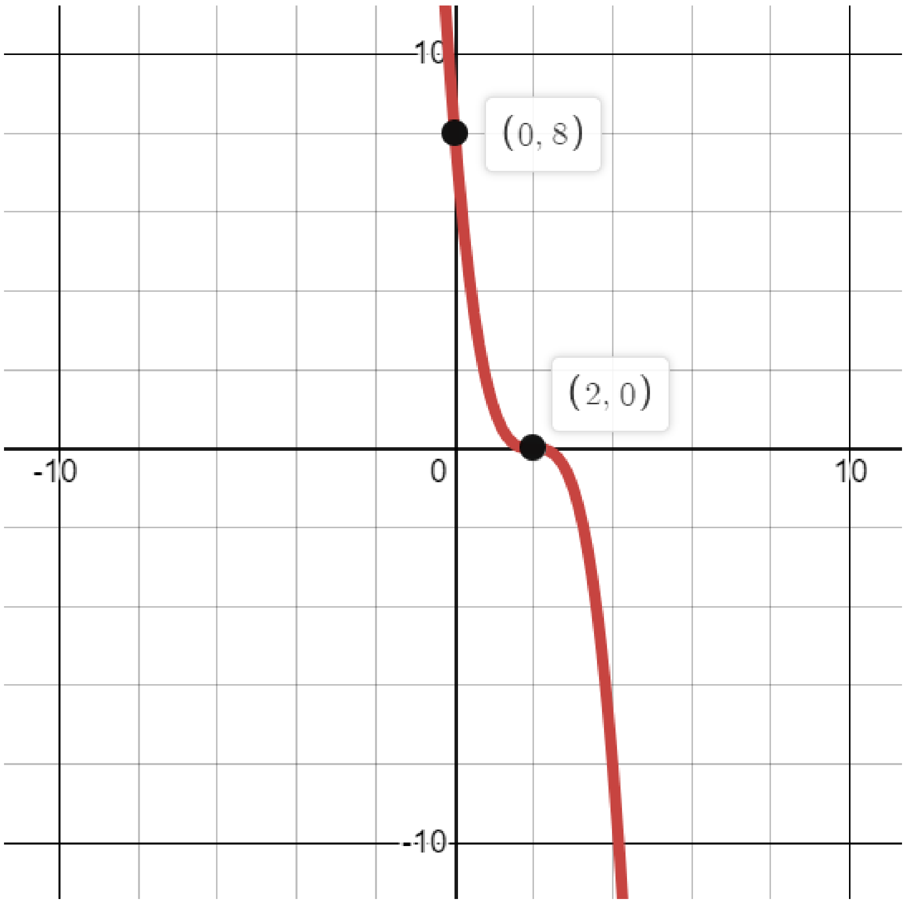
Worked example	Your turn
Sketch the graph of: $$y = (x + 4)^3$$	Sketch the graph of: $$y = (x - 2)^3$$

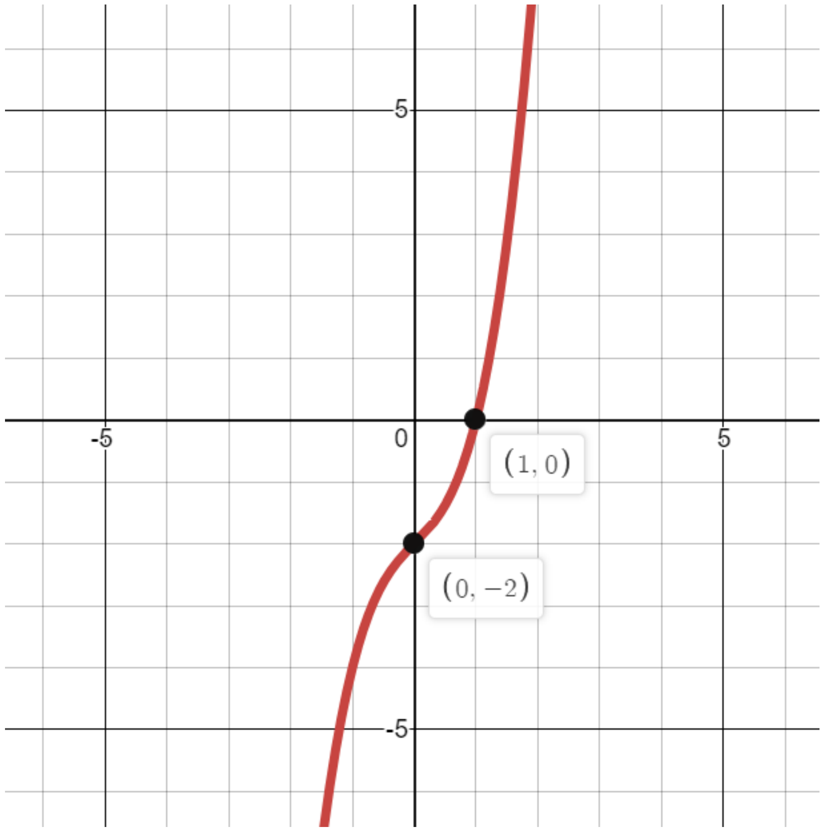


Worked example	Your turn
Sketch the graph of: $y = -(x+4)^3$	Sketch the graph of: $y = -(x-2)^3$

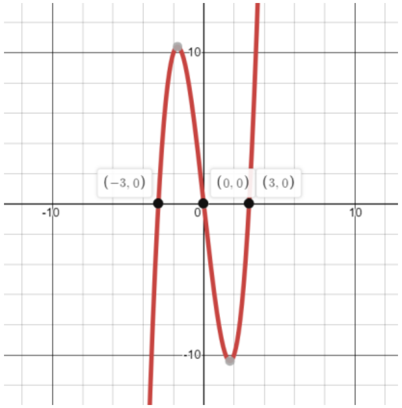


Worked example	Your turn
Sketch the graph of: $y = (4 - x)^3$	Sketch the graph of: $y = (2 - x)^3$

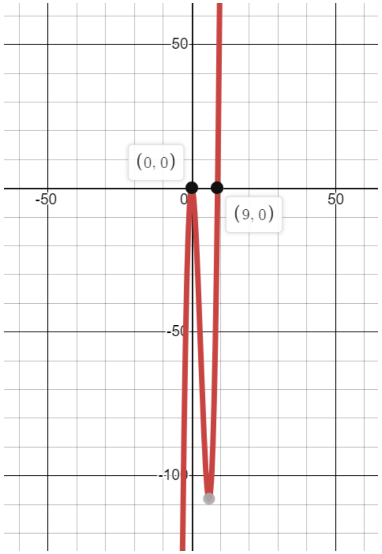


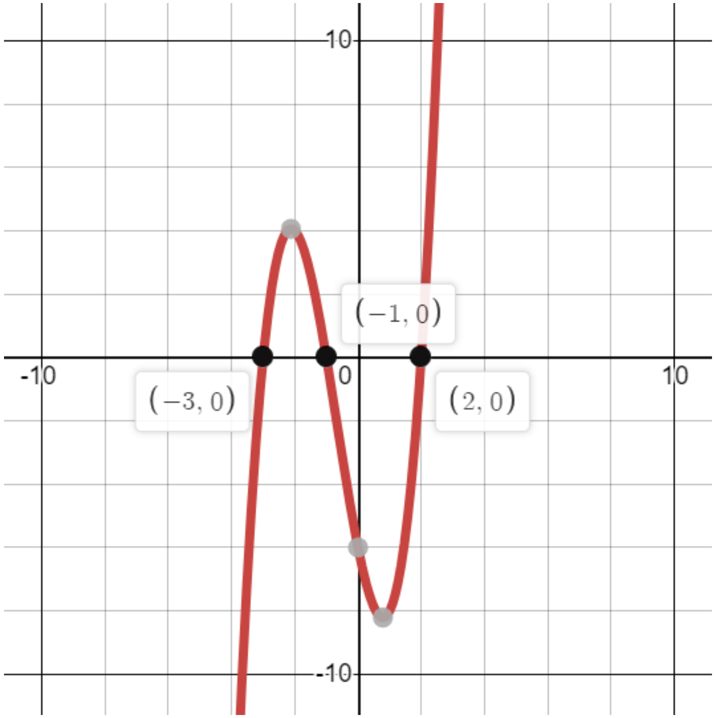
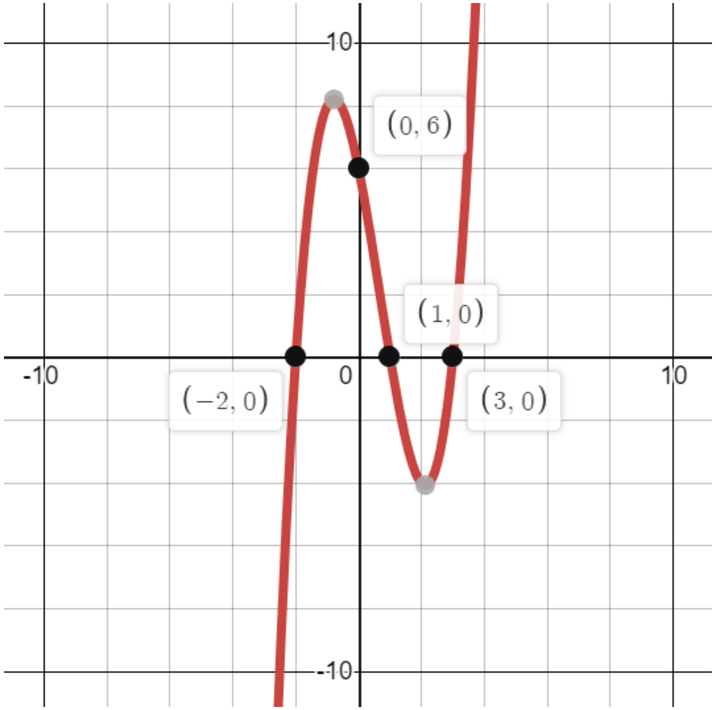
Worked example	Your turn
Sketch the graph of: $y=(x+2)(x^2+2x+4)$	Sketch the graph of: $y=(x-1)(x^2+x+2)$ 

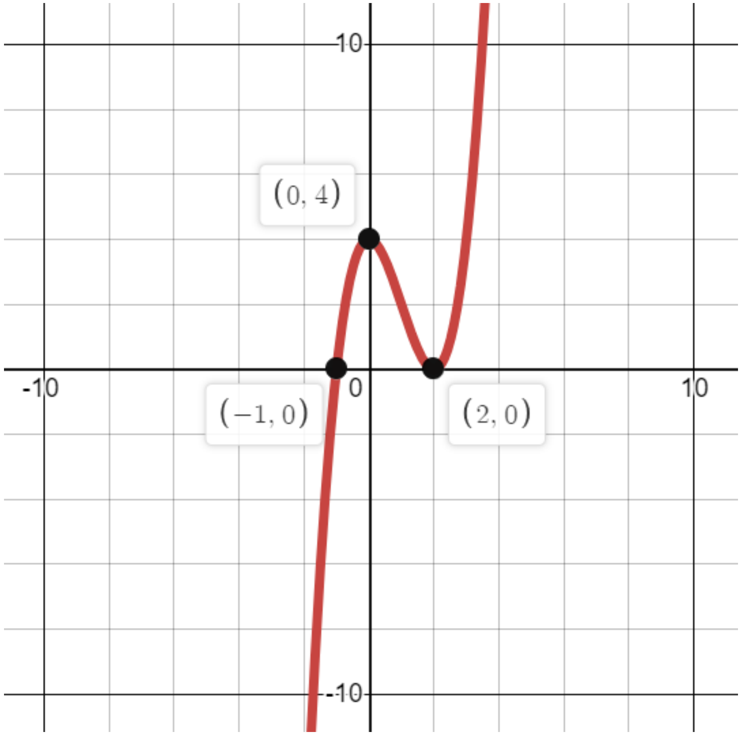
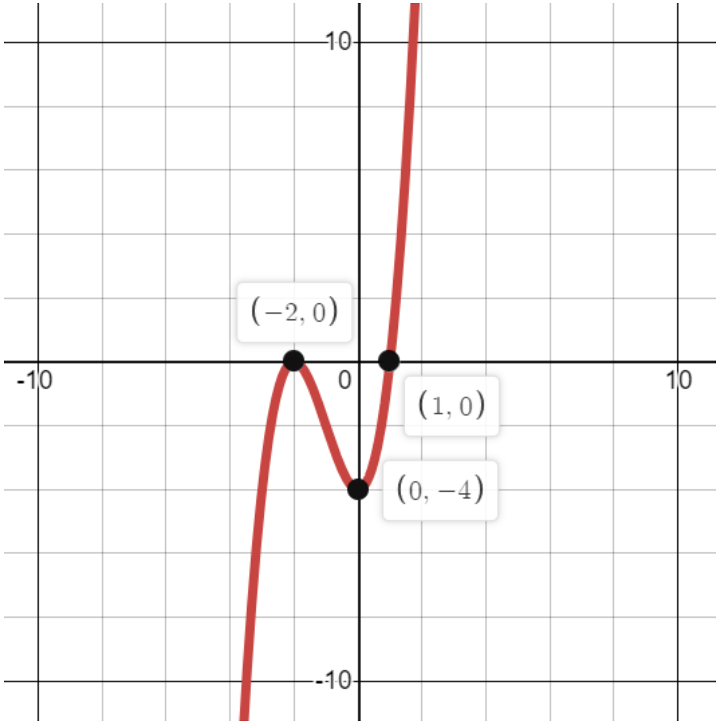
Worked example	Your turn
Sketch the graphs of: $y = x^3 - 16x$	Sketch the graphs of: $y = x^3 - 9x$



$y = x^3 - 16x^2$	$y = x^3 - 9x^2$



Worked example	Your turn
The graph of $y = ax^3 + bx^2 + cx + d$ is shown where $a, b, c, d \in \mathbb{R}$. Find the value of a, b, c and d	The graph of $y = ax^3 + bx^2 + cx + d$ is shown where $a, b, c, d \in \mathbb{R}$. Find the value of a, b, c and d
	
	$a = 1, b = -2, c = -5, d = 6$

Worked example	Your turn
The graph of $y = ax^3 + bx^2 + cx + d$ is shown where $a, b, c, d \in \mathbb{R}$. Find the value of a, b, c and d	The graph of $y = ax^3 + bx^2 + cx + d$ is shown where $a, b, c, d \in \mathbb{R}$. Find the value of a, b, c and d
	
	$a = 1, b = 3, c = 0, d = -4$

Worked example	Your turn
A curve is a positive cubic, touches the x-axis at 3 and crosses the x-axis at -2. Write a possible equation for the curve.	A curve is a positive cubic, touches the x-axis at 3 and crosses the x-axis at -2. Write a possible equation for the curve. $y=(x-3)^2(x+2)$

4.2) Quartic graphs

Chapter CONTENTS

Worked example

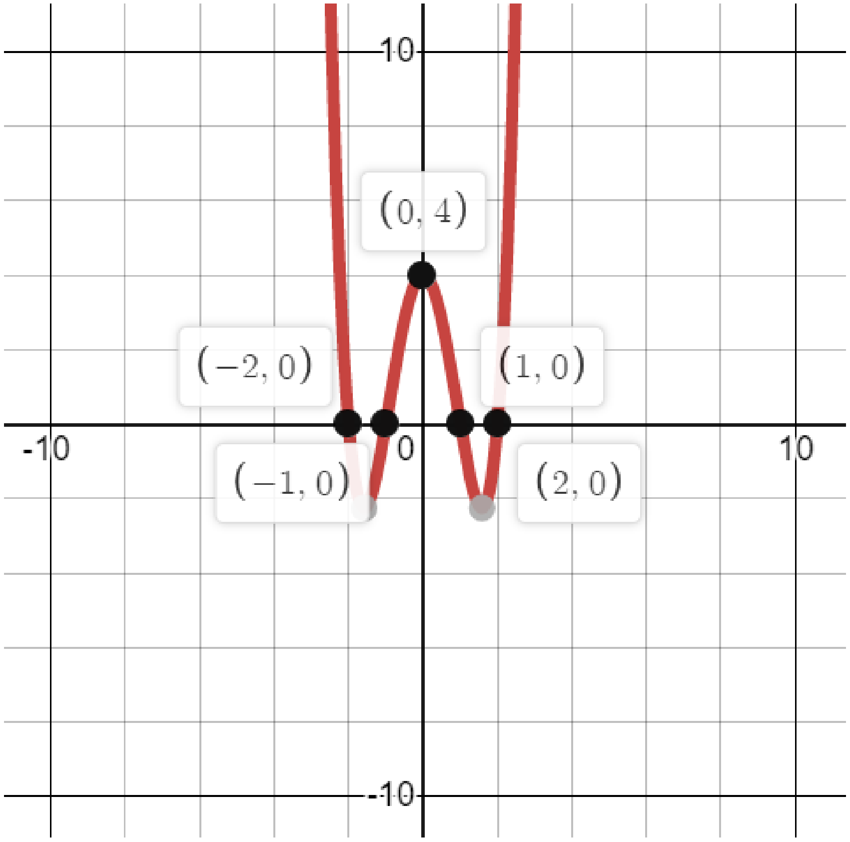
Sketch the graph of:

$$y = (x+3)(x+4)(x-3)(x-4)$$

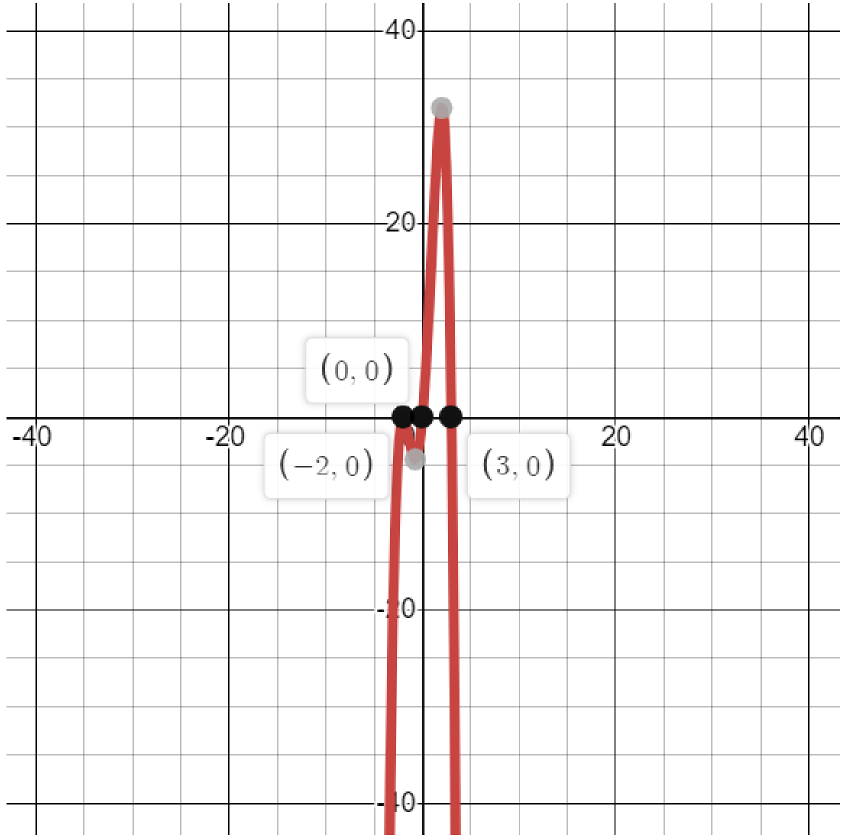
Your turn

Sketch the graph of:

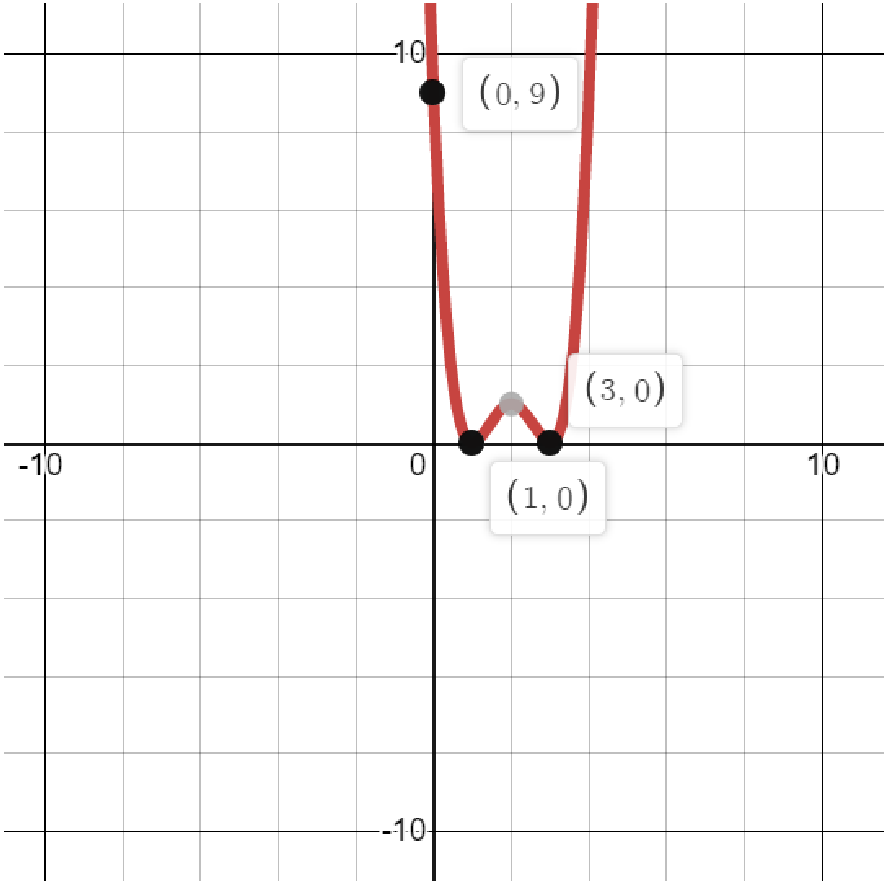
$$y = (x+1)(x+2)(x-1)(x-2)$$



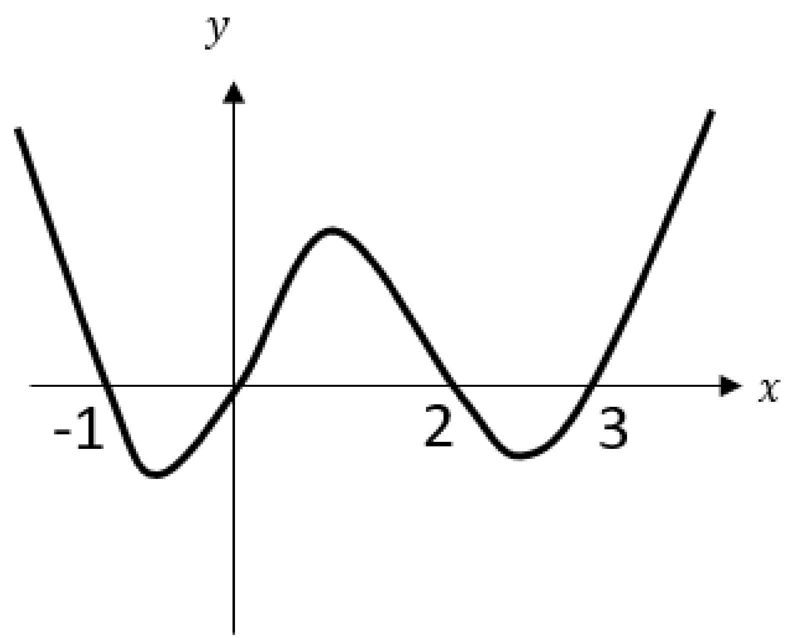
Worked example	Your turn
Sketch the graph of: $y = x(x-3)^2(2-x)$	Sketch the graph of: $y = x(x+2)^2(3-x)$



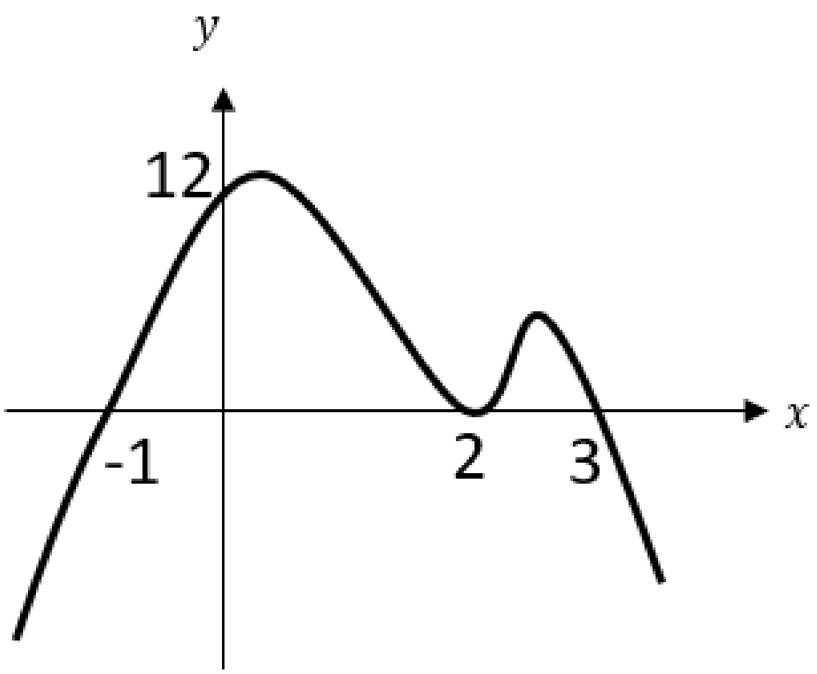
Worked example	Your turn
Sketch the graph of: $y = (x+2)^2(x-4)^2$	Sketch the graph of: $y = (x-1)^2(x-3)^2$



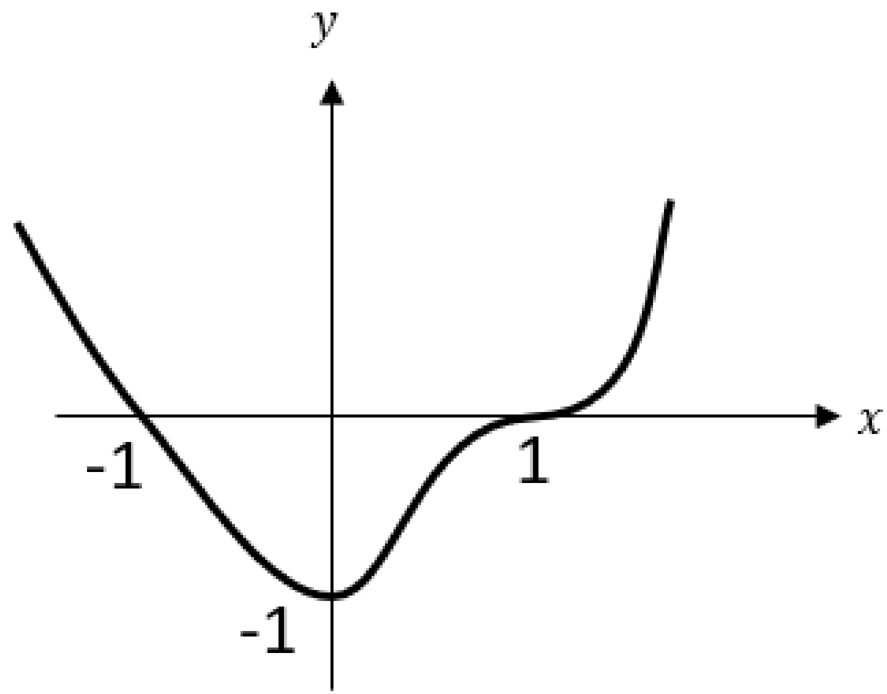
Worked example	Your turn
Sketch the graph of: $y = x(x-4)(x+5)(x+6)$	Sketch the graph of: $y = x(x+1)(x-2)(x-3)$



Worked example	Your turn
Sketch the graph of: $y = (x+4)^2(x-5)(6-x)$	Sketch the graph of: $y = (x-2)^2(x+1)(3-x)$



Worked example	Your turn
Sketch the graph of: $y = (x-2)(x+2)^3$	Sketch the graph of: $y = (x+1)(x-1)^3$



Worked example

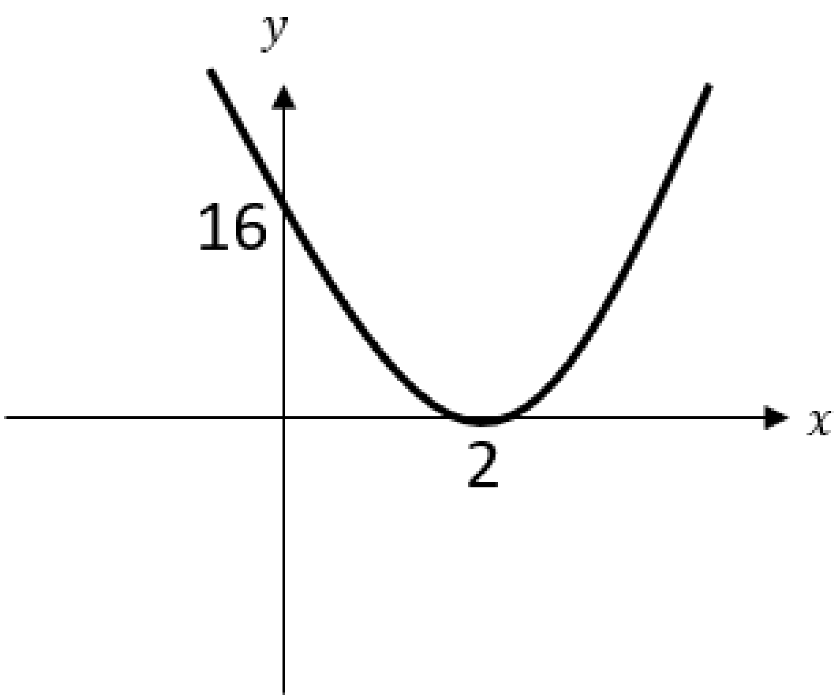
Sketch the graph of:

$$y = (x+3)^4$$

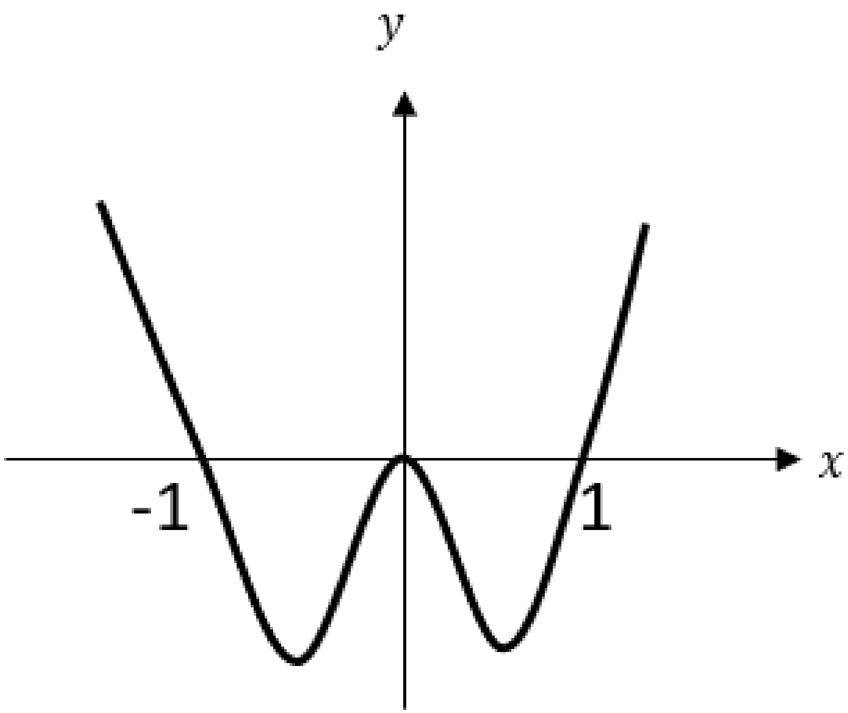
Your turn

Sketch the graph of:

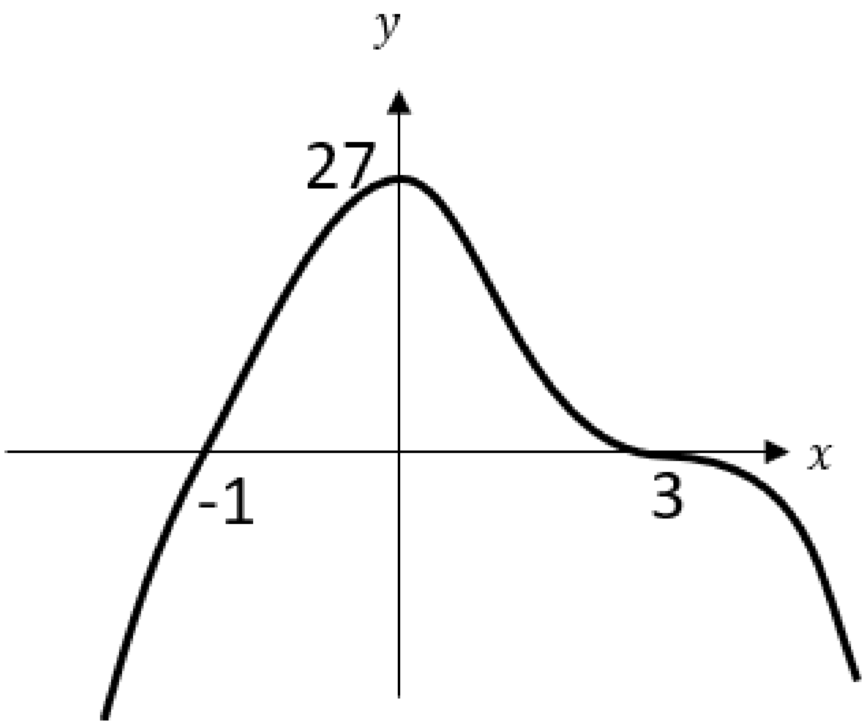
$$y = (x-2)^4$$



Worked example	Your turn
Sketch the graph of: $y = x^2(x+2)(x-2)$	Sketch the graph of: $y = x^2(x+1)(x-1)$

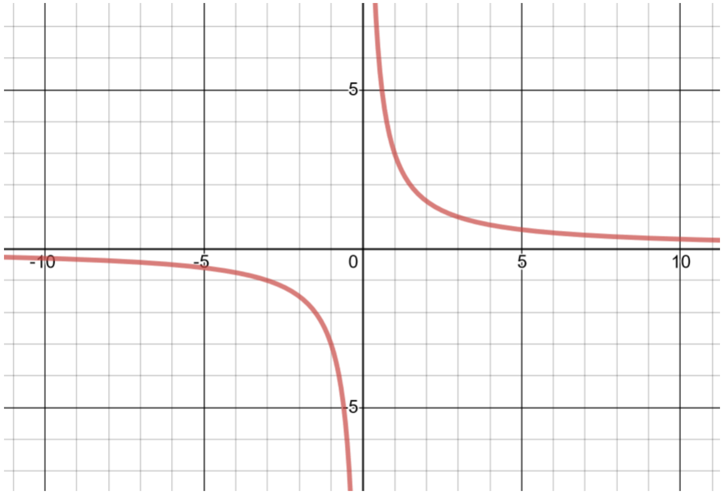


Worked example	Your turn
Sketch the graph of: $$y = -(x-4)(x+2)^3$$	Sketch the graph of: $$y = -(x+1)(x-3)^3$$

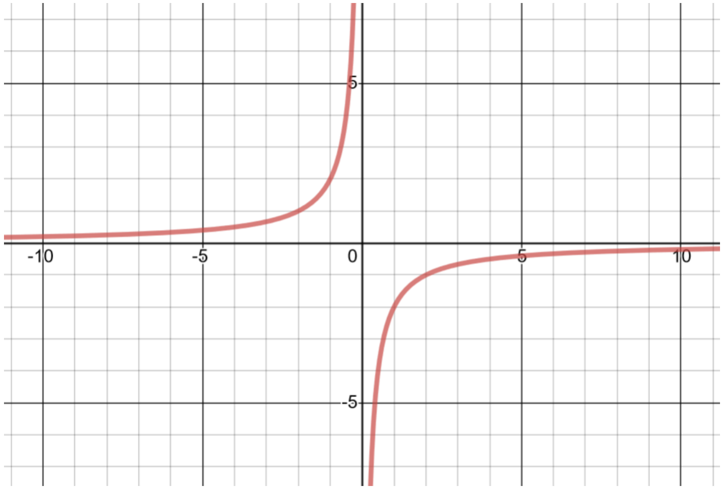


4.3) Reciprocal graphs

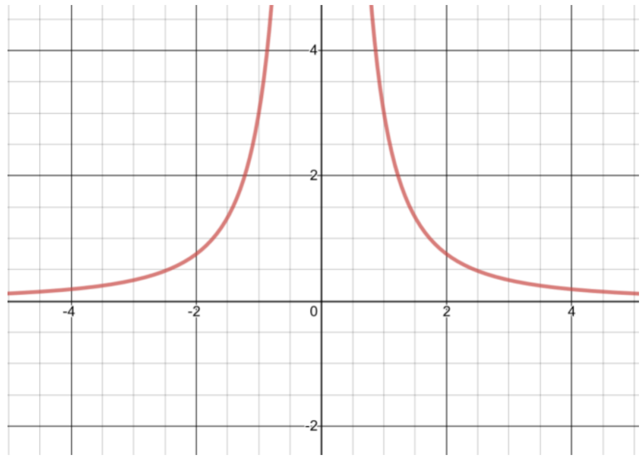
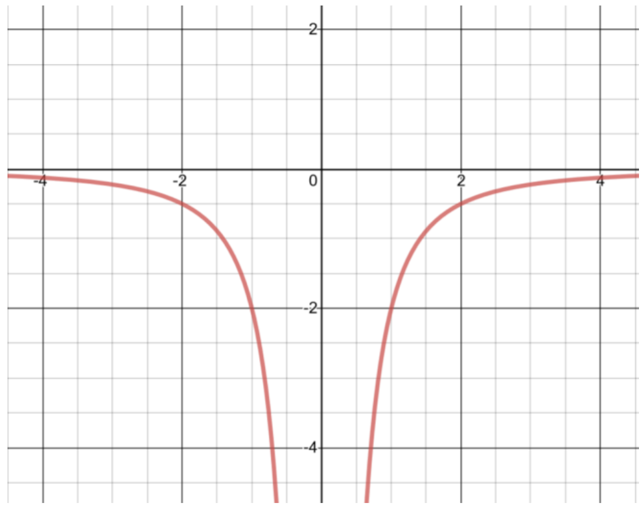
Worked example	Your turn
Sketch the graph of $y = \frac{a}{x}, a > 0$	Sketch the graph of $y = \frac{3}{x}$



Sketch the graph of $y = \frac{a}{x}, a < 0$	Sketch the graph of $y = \frac{-2}{x}$

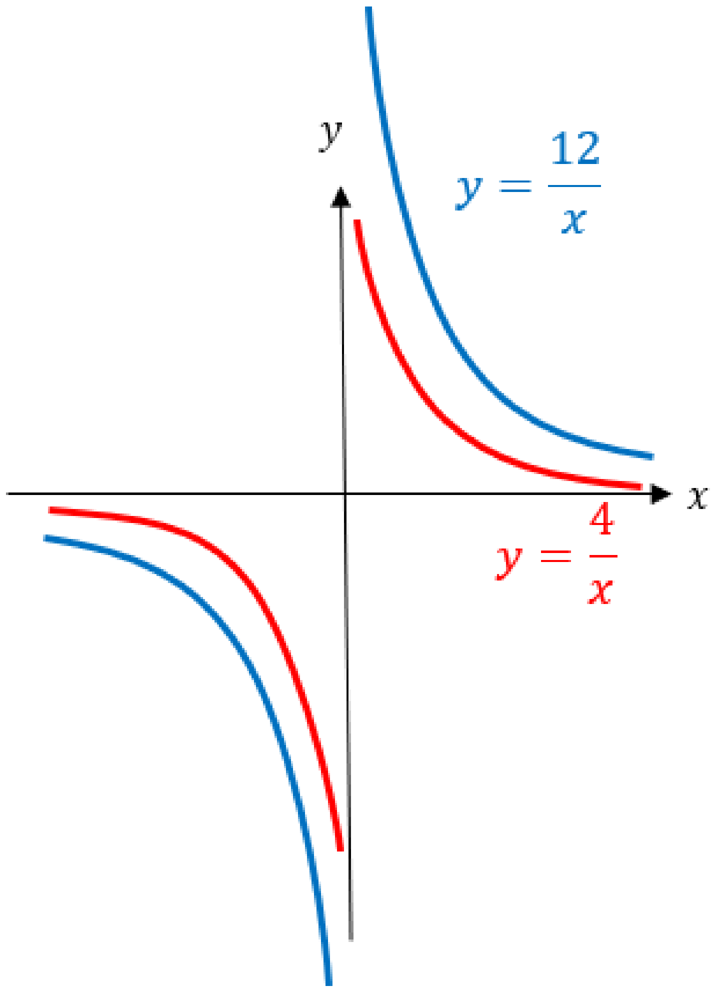


Graphs used with permission from DESMOS: https://www.desmos.com/

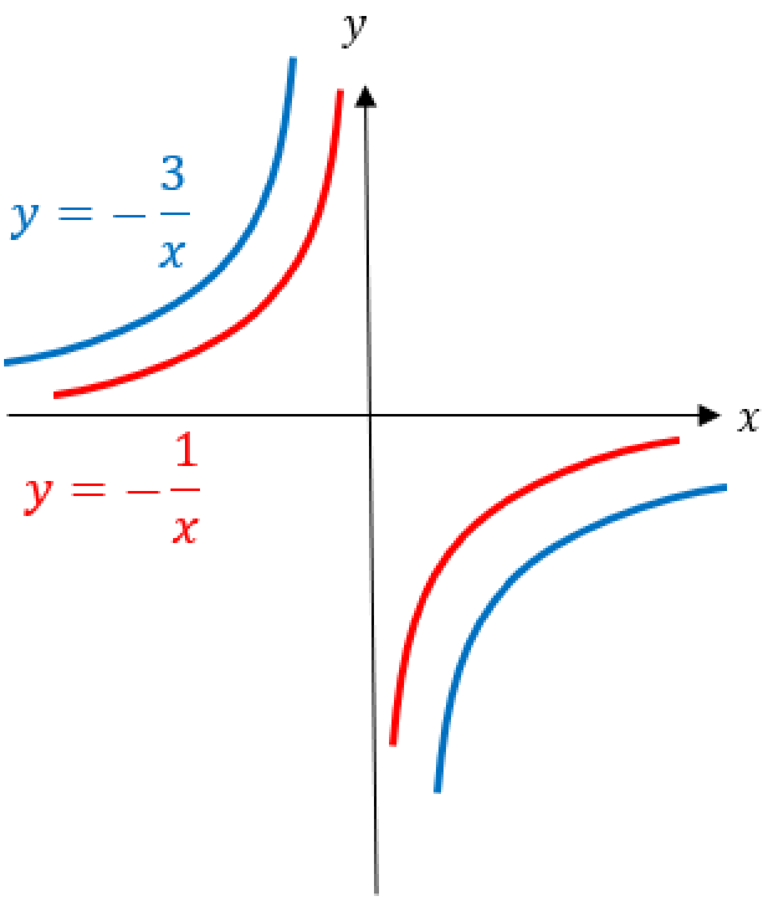
Worked example	Your turn
Sketch the graph of $y = \frac{a}{x^2}, a > 0$	Sketch the graph of $y = \frac{3}{x^2}$ 
Sketch the graph of $y = \frac{a}{x^2}, a < 0$	Sketch the graph of $y = \frac{-2}{x^2}$ 

Graphs used with permission from DESMOS: https://www.desmos.com/

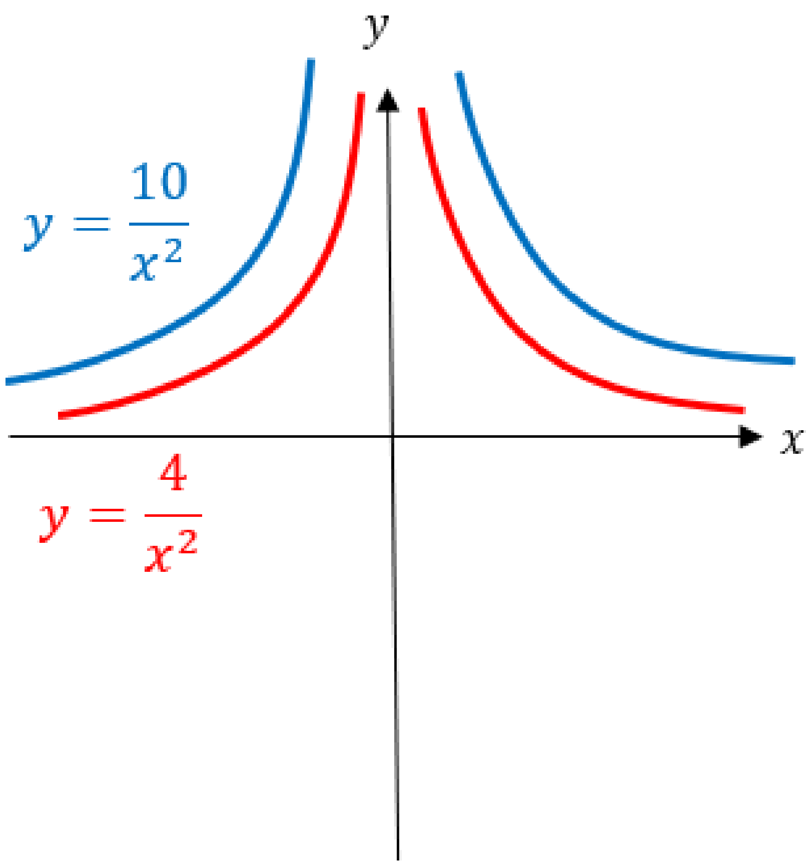
Worked example	Your turn
Sketch on the same diagram: $y = \frac{2}{x}$ and $y = \frac{8}{x}$	Sketch on the same diagram: $y = \frac{4}{x}$ and $y = \frac{12}{x}$



Worked example	Your turn
Sketch on the same diagram: $y = -\frac{2}{x}$ and $y = -\frac{8}{x}$	Sketch on the same diagram: $y = -\frac{1}{x}$ and $y = -\frac{3}{x}$



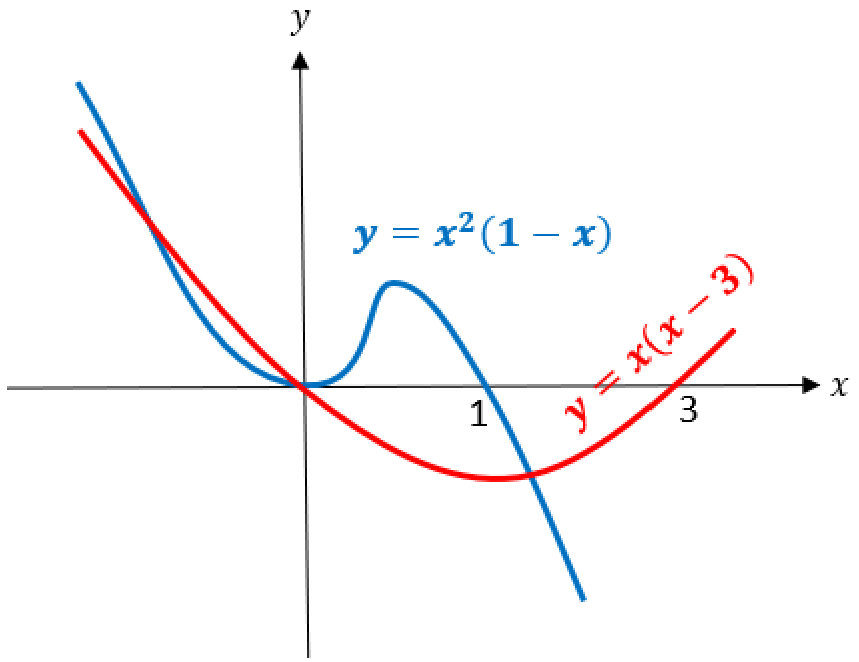
Worked example	Your turn
Sketch on the same diagram: $y = \frac{2}{x^2}$ and $y = \frac{7}{x^2}$	Sketch on the same diagram: $y = \frac{4}{x^2}$ and $y = \frac{10}{x^2}$



4.4) Points of intersection

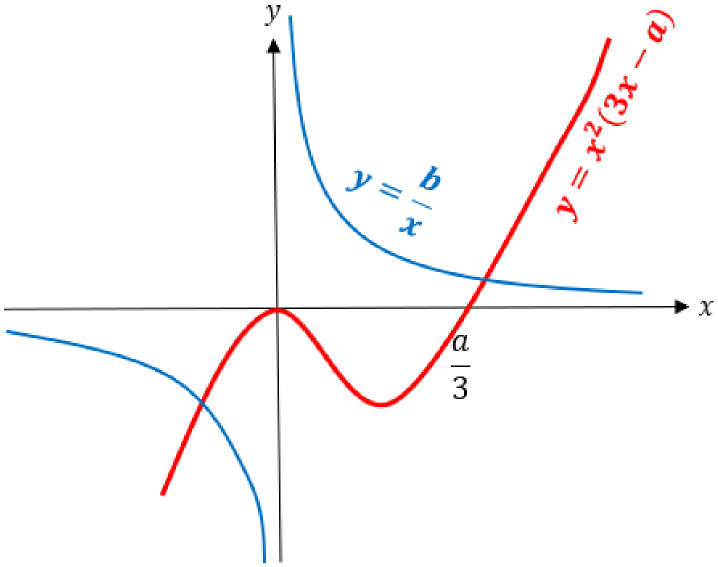
Chapter CONTENTS

Worked example	Your turn
On the same diagram sketch the curves with equations $y = x(x-2)$ and $y = x^2(1-x)$. Find the coordinates of their points of intersection.	On the same diagram sketch the curves with equations $y = x(x-3)$ and $y = x^2(1-x)$. Find the coordinates of their points of intersection.



$\left(-\sqrt{3}, 3+3\sqrt{3}\right)$,
$(0,0)$,
$\left(\sqrt{3}, 3-3\sqrt{3}\right)$

Worked example	Your turn
On the same diagram sketch the curves with equations $y = -x^2(5x - a)$ and $y = -\frac{b}{x}$, where a, b are positive constants. State, giving a reason, the number of real solutions to the equation $x^2(5x - a) + \frac{b}{x} = 0$	On the same diagram sketch the curves with equations $y = x^2(3x - a)$ and $y = \frac{b}{x}$, where a, b are positive constants. State, giving a reason, the number of real solutions to the equation $x^2(3x - a) - \frac{b}{x} = 0$



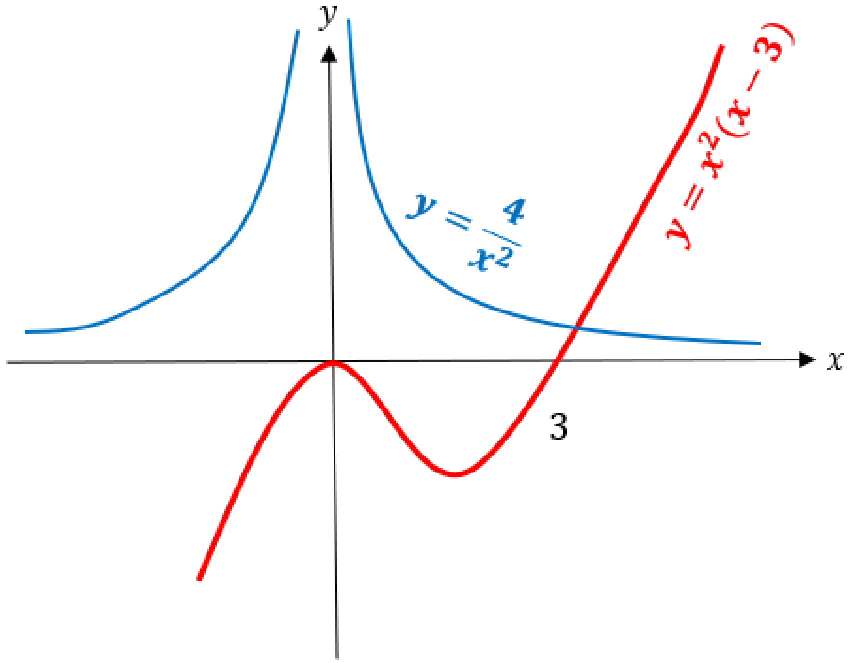
2 points of intersection where

$$x^2(3x - a) = \frac{b}{x}$$

$$x^2(3x - a) - \frac{b}{x} = 0$$

∴ 2 solutions

Worked example	Your turn
On the same diagram sketch the curves with equations $y = \frac{3}{x^2}$ and $y = x^2(x-4)$. State, giving a reason, the number of real solutions to the equation $x^4(x-4) - 3 = 0$	On the same diagram sketch the curves with equations $y = \frac{4}{x^2}$ and $y = x^2(x-3)$. State, giving a reason, the number of real solutions to the equation $x^4(x-3) - 4 = 0$

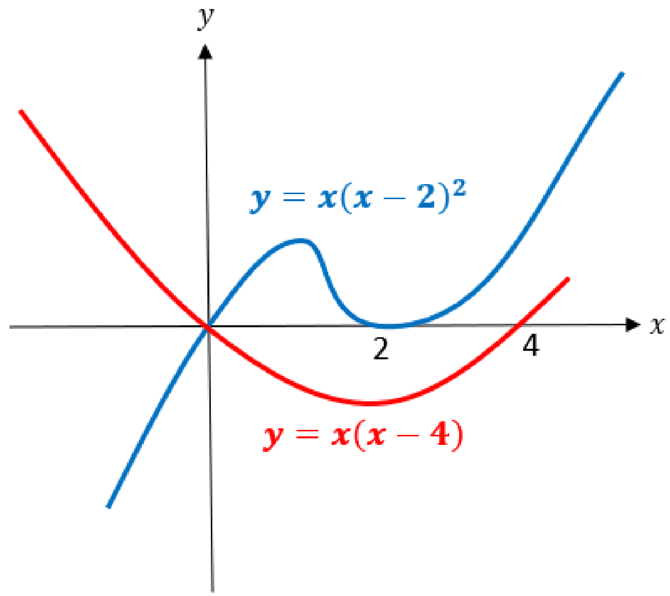


1 point of intersection where

$$x^2(x-3) = \frac{4}{x^2}$$
$$x^4(x-3) = 4$$
$$x^4(x-3) - 4 = 0$$

$\therefore$ 1 real solution

Worked example	Your turn
On the same diagram sketch the curves with equations $y = x(x-5)$ and $y = x(x-3)^2$, and hence find the coordinates of any points of intersection.	On the same diagram sketch the curves with equations $y = x(x-4)$ and $y = x(x-2)^2$, and hence find the coordinates of any points of intersection.



$(0, 0)$ only as:

$$x(x-2)^2 = x(x-4)$$
$$x(x^2-4x+4) = x^2-4x$$
$$x^3-4x^2+4x = x^2-4x$$
$$x^3-5x^2+8x = 0$$
$$x(x^2-5x+8) = 0$$

Discriminant of $x^2-5x+8 = -7 < 0$

Worked example	Your turn
Work out the range of values of a such that the graphs of $y = x^2 + a$ and $3y = x - 2$ have two points of intersection	Work out the range of values of a such that the graphs of $y = x^2 + a$ and $4y = x - 3$ have two points of intersection $a < -\frac{47}{72}$

4.5) Translating graphs

Worked example	Your turn
Describe the effect on the graph of $y = f(x)$ of:	Describe the effect on the graph of $y = f(x)$ of:
$f(x+9)$	$f(x+2)$ Translation by vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
$f(x-8)$	$f(x-3)$ Translation by vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$
$f(x)+7$	$f(x)+4$ Translation by vector $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$
$f(x)-6$	$f(x)-5$ Translation by vector $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$

Worked example	Your turn
Sketch:	Sketch:

Worked example

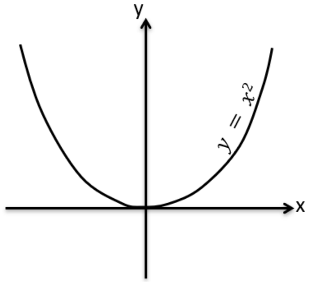
$$y = -x^2$$

$$y = -x^2 - 3$$

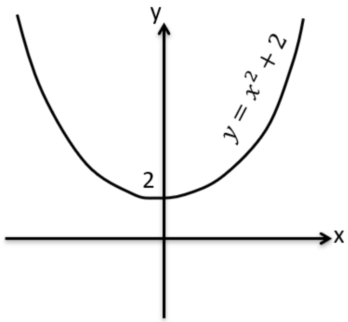
$$y = -(x-3)^2$$

Your turn

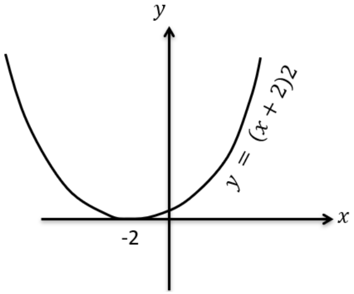
$$y = x^2$$



$$y = x^2 + 2$$



$$y = (x+2)^2$$



Worked example	Your turn
$f(x) = -x^3$	$g(x) = x^3$

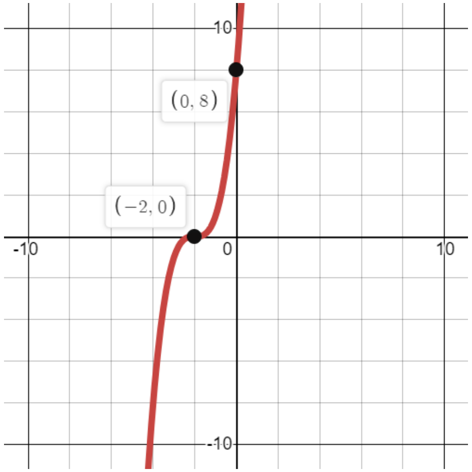
Sketch:

$f(x-3)$

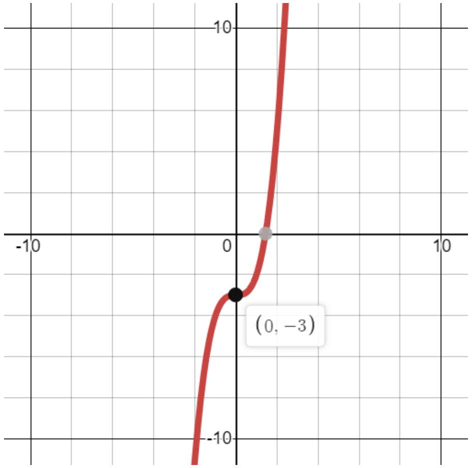
$f(x)+2$

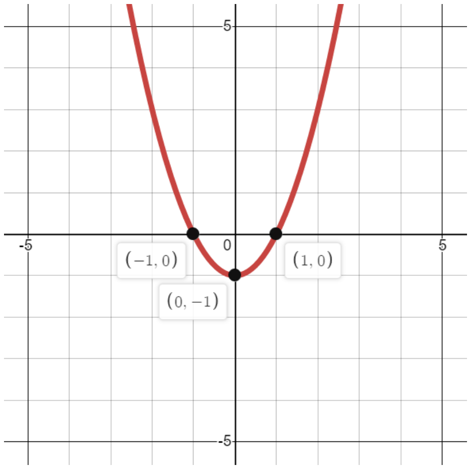
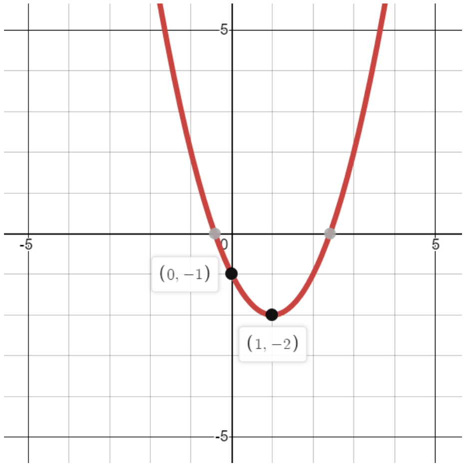
Sketch:

$g(x+2)$



$g(x)-3$



Worked example	Your turn
$f(x) = x(x+3)$	$g(x) = x(x-2)$
Sketch:	Sketch:
$f(x-3)$	$g(x+1)$
	
$f(x)+2$	$g(x)-1$
	

Worked example	Your turn
$$f(x) = -\frac{2}{x}$$	$$g(x) = \frac{3}{x}$$

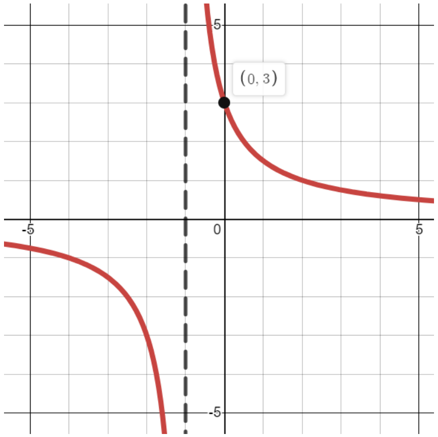
Sketch:

$$f(x-3)$$

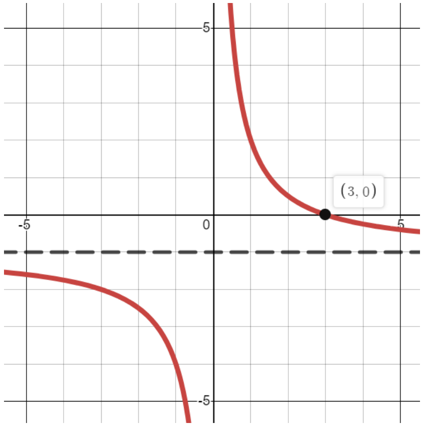
$$f(x)+2$$

Sketch:

$$g(x+1)$$

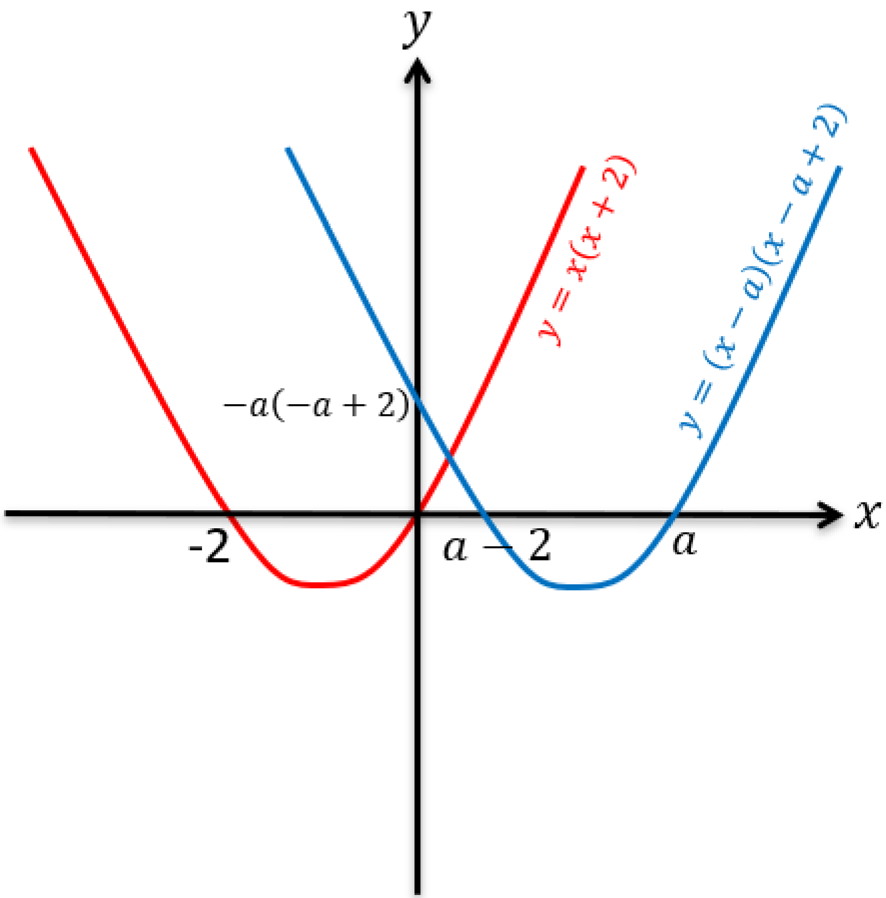


$$g(x)-1$$



Worked example	Your turn
The point with coordinates $(-1.5, 0)$ lies on the curve with equation $y = (x+a)^3 + 6(x+a)^2 + 9(x+a)$ where a is a constant. Find the two possible values of a	The point with coordinates $(-2, 0)$ lies on the curve with equation $y = (x+a)^3 + 8(x+a)^2 + 16(x+a)$ where a is a constant. Find the two possible values of a $a = \pm 2$

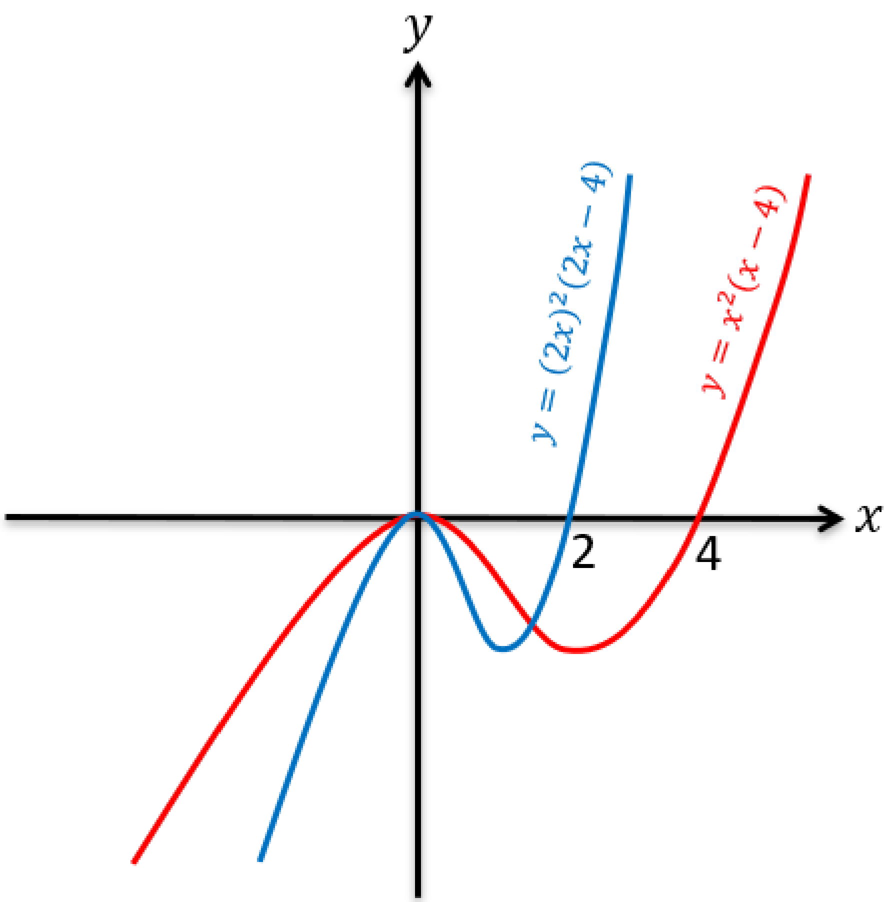
Worked example	Your turn
Sketch $y = x(x-3)$. On the same axes, sketch $y = (x+a)(x+a-3)$, where $a > 3$.	Sketch $y = x(x+2)$. On the same axes, sketch $y = (x-a)(x-a+2)$, where $a > 2$.

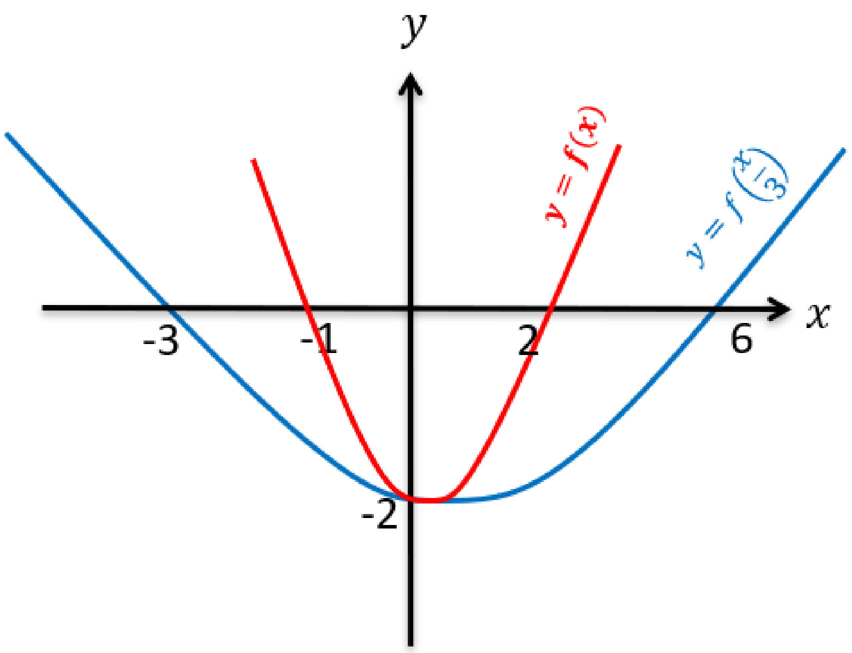


4.6) Stretching graphs

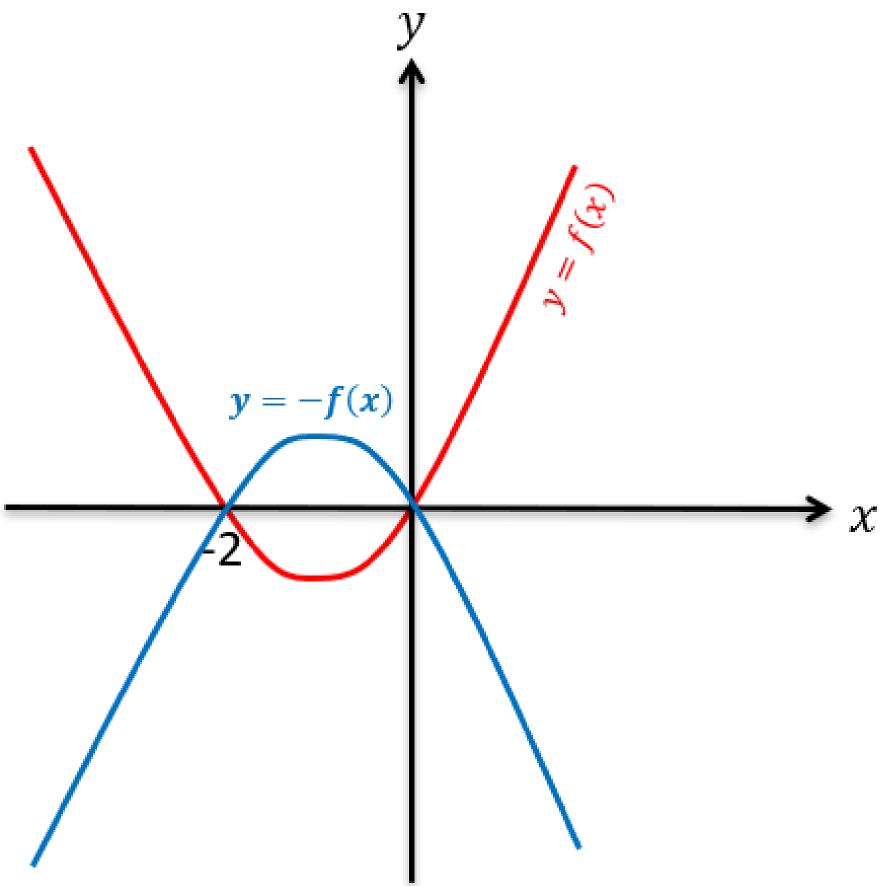
Worked example	Your turn
Describe the effect on the graph of $y = f(x)$ of:	Describe the effect on the graph of $y = f(x)$ of:
$f(9x)$	$f(2x)$ Stretch, scale factor $\frac{1}{2}$, in the x-direction
$f(\frac{1}{8}x)$	$f(\frac{1}{3}x)$ Stretch, scale factor 3, in the x-direction
$7f(x)$	$4f(x)$ Stretch, scale factor 4, in the y-direction
$\frac{1}{6}f(x)$	$\frac{1}{5}f(x)$ Stretch, scale factor $\frac{1}{5}$, in the y-direction

Worked example	Your turn
Sketch $y = x^2(x+8)$. On the same axes, sketch the graph with equation $y = (4x)^2(4x+8)$	Sketch $y = x^2(x-4)$. On the same axes, sketch the graph with equation $y = (2x)^2(2x-4)$

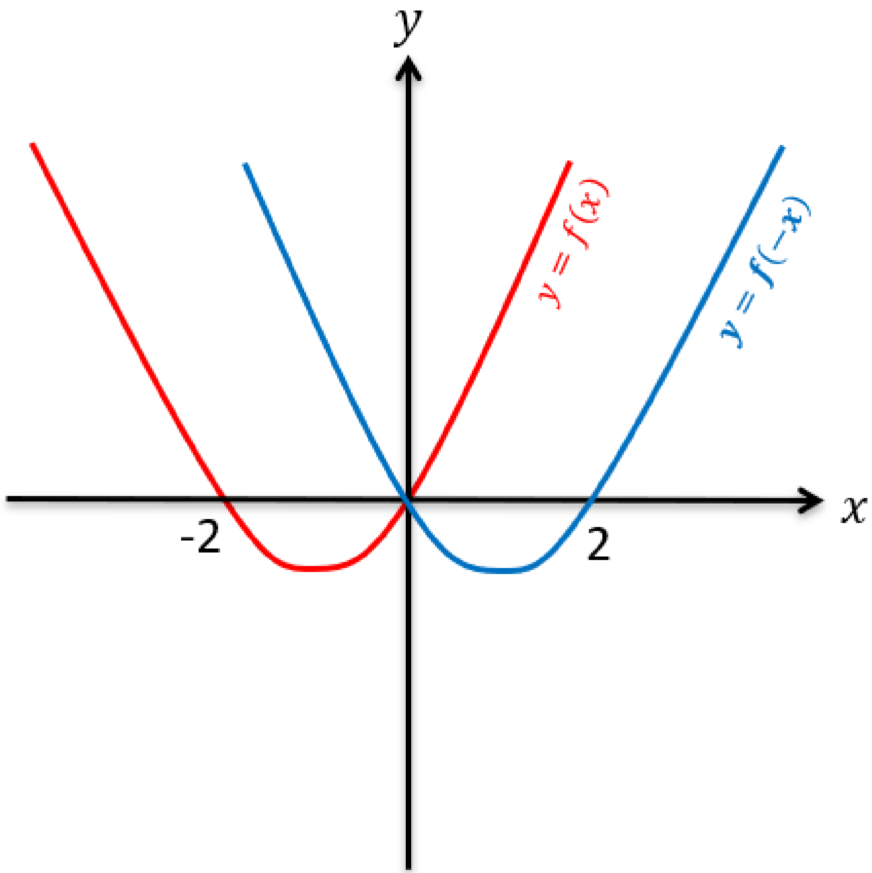


Worked example	Your turn
If $y = (x+2)(x-1)$, sketch $y = f(x)$ and $y = f(\frac{x}{4})$ on the same axes.	If $y = (x+1)(x-2)$, sketch $y = f(x)$ and $y = f(\frac{x}{3})$ on the same axes. 

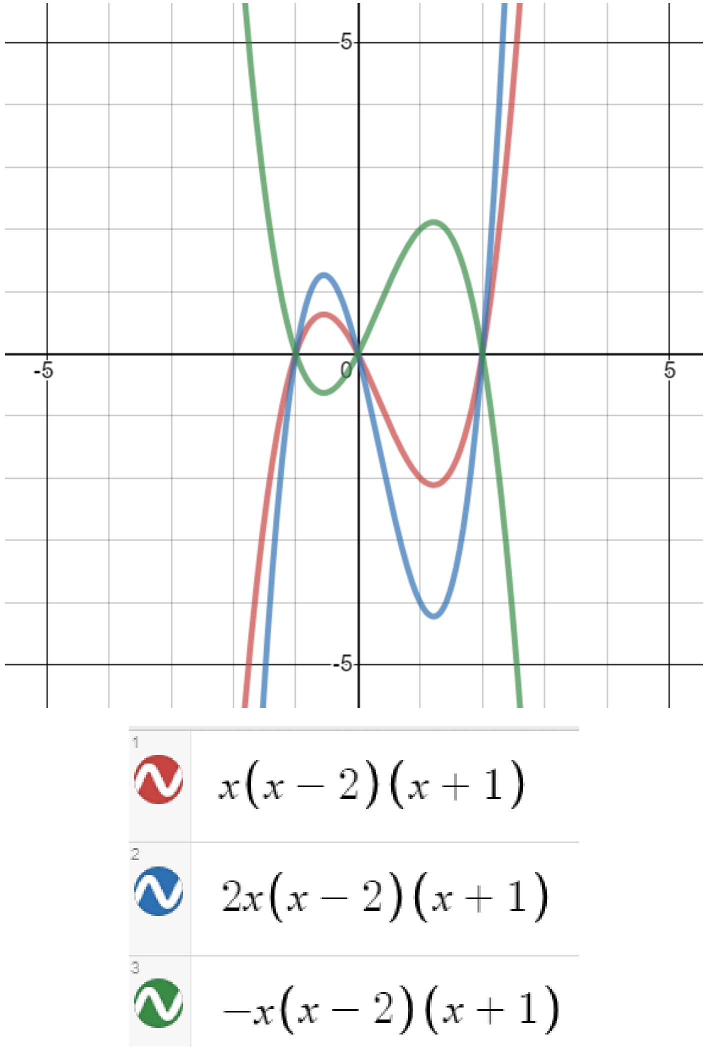
Worked example	Your turn
If $y = x(x-3)$, sketch $y = f(x)$ and $y = -f(x)$ on the same axes.	If $y = x(x+2)$, sketch $y = f(x)$ and $y = -f(x)$ on the same axes.



Worked example	Your turn
If $y = x(x-3)$, sketch $y = f(x)$ and $y = f(-x)$ on the same axes.	If $y = x(x+2)$, sketch $y = f(x)$ and $y = f(-x)$ on the same axes.

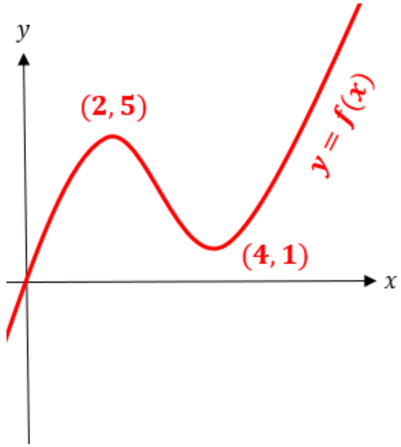


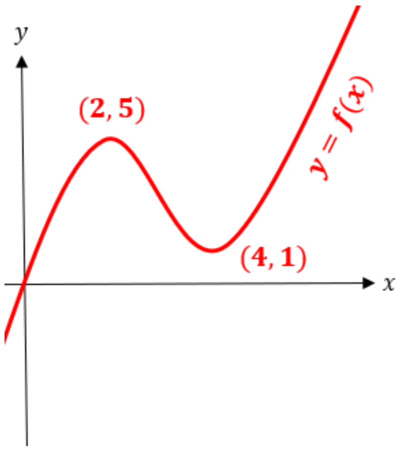
Worked example	Your turn
On the same axes, sketch:	On the same axes, sketch:
$y = x(x+2)(x-1)$	$y = x(x-2)(x+1)$
$y = 4x(4x+2)(4x-1)$	$y = 2x(2x-2)(2x+1)$
$y = -x(x+2)(x-1)$	$y = -x(x-2)(x+1)$



4.7) Transforming functions

Worked example	Your turn
A sketch of the curve $y = f(x)$ is shown.	A sketch of the curve $y = f(x)$ is shown.



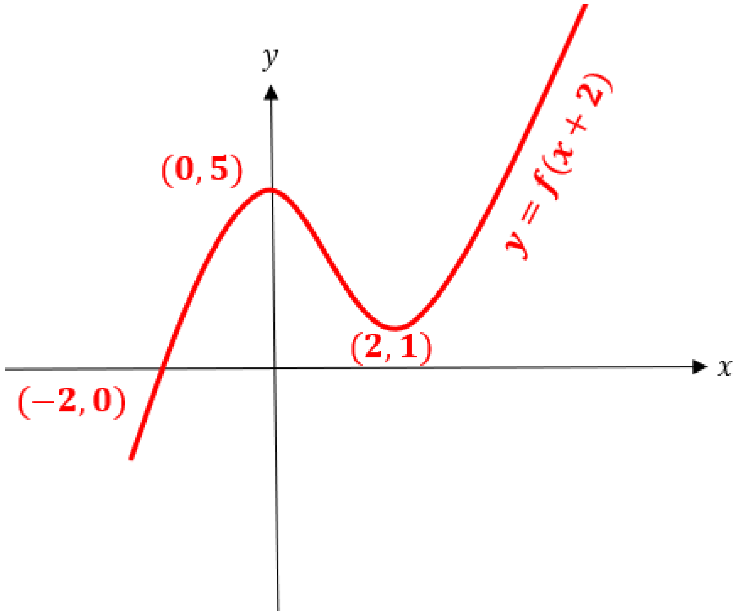


Sketch:

$$y = f(x - 3)$$

Sketch:

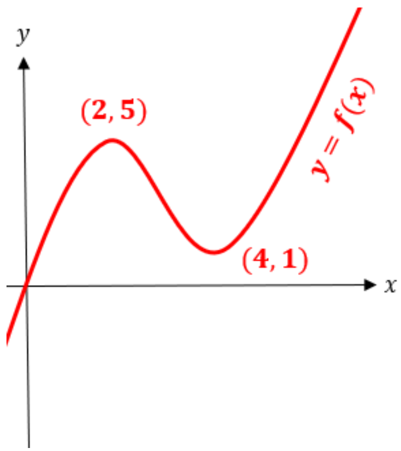
$$y = f(x + 2)$$



Worked example	Your turn
A sketch of the curve $y = f(x)$ is shown.	A sketch of the curve $y = f(x)$ is shown.

Worked example

A sketch of the curve $y = f(x)$ is shown.

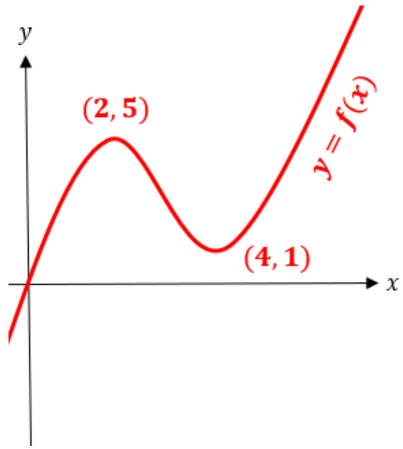


Sketch:

$$y = f(x) - 3$$

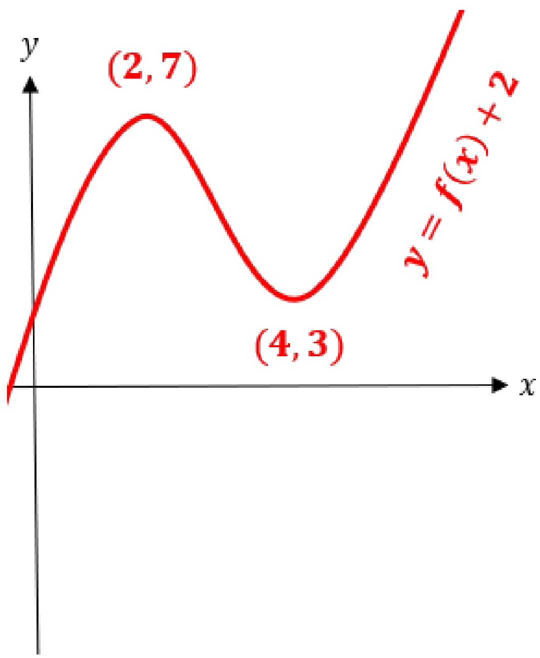
Your turn

A sketch of the curve $y = f(x)$ is shown.

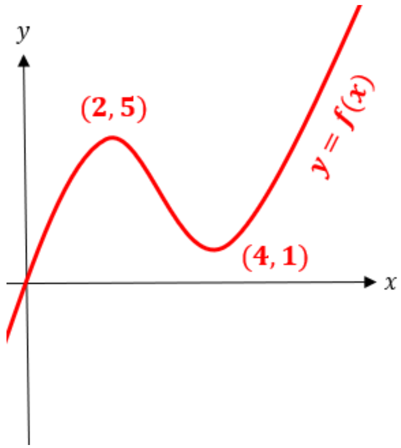


Sketch:

$$y = f(x) + 2$$

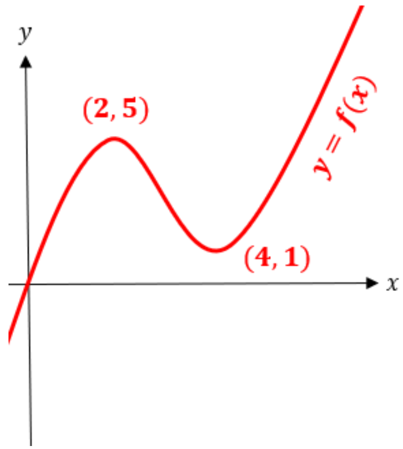


Worked example	Your turn
A sketch of the curve $y = f(x)$ is shown.	A sketch of the curve $y = f(x)$ is shown.



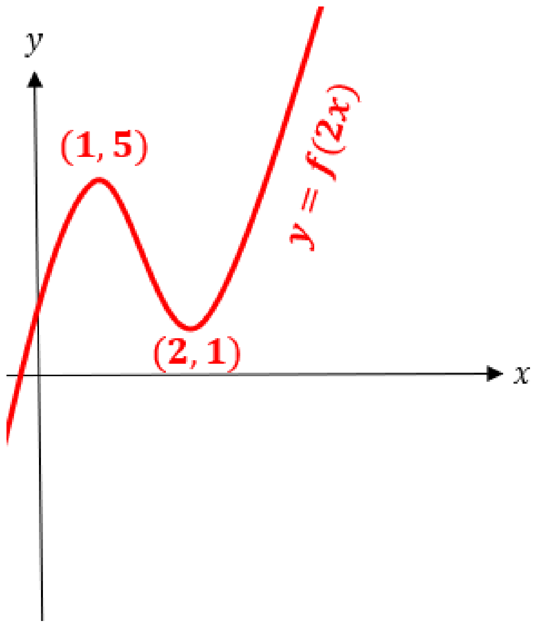
Sketch:

$$y = f(4x)$$

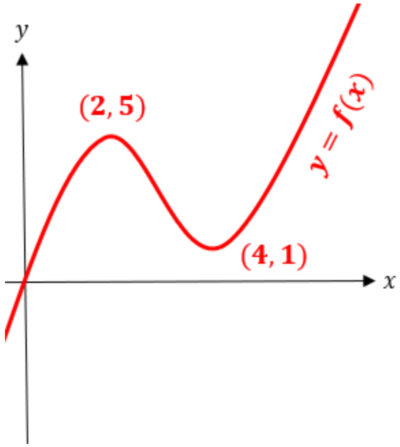


Sketch:

$$y = f(2x)$$



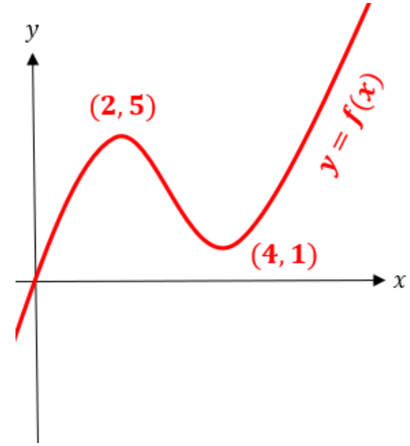
Worked example	Your turn
A sketch of the curve $y = f(x)$ is shown.	A sketch of the curve $y = f(x)$ is shown.



Sketch:

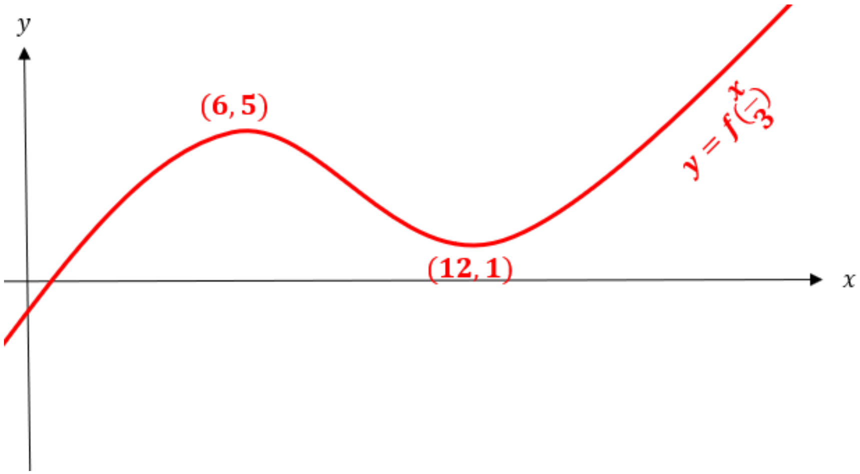
$$y = f\left(\frac{x}{4}\right)$$

A sketch of the curve $y = f(x)$ is shown.

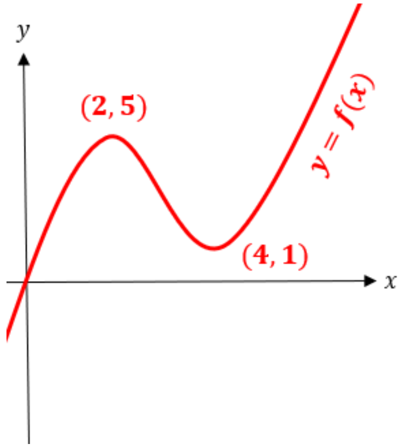


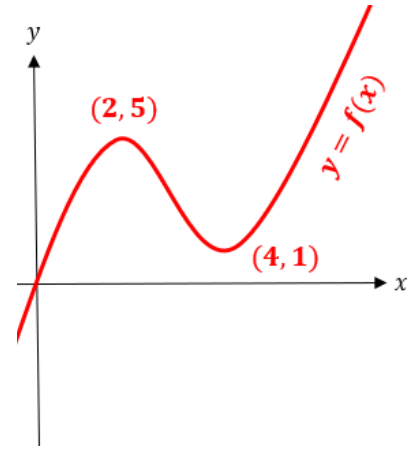
Sketch:

$$y = f\left(\frac{x}{3}\right)$$



Worked example	Your turn
A sketch of the curve $y = f(x)$ is shown.	A sketch of the curve $y = f(x)$ is shown.



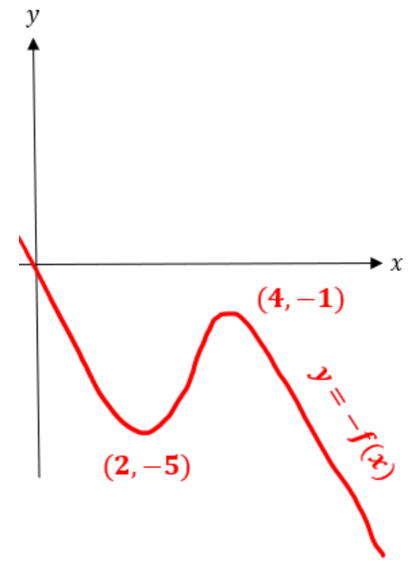


Sketch:

$$y = f(-x)$$

Sketch:

$$y = -f(x)$$



Worked example

Find the new coordinates under the transformations

$y = f(x)$	$(-6, 4)$	$(0, 1)$
$y = f(x+2)$		
$y = f(x) - 2$		
$y = f(3x)$		
$y = 4f(x)$		
$y = f\left(\frac{x}{5}\right)$		
$y = 6f(x)$		
$y = -f(x)$		
$y = f(-x)$		

Your turn

Find the new coordinates under the transformations

$y = f(x)$	$(6, -4)$	$(1, 0)$
$y = f(x+1)$	$(5, -4)$	$(0, 0)$
$y = f(x) - 1$	$(6, -5)$	$(1, -1)$
$y = f(2x)$	$(3, -4)$	$(\frac{1}{2}, 0)$
$y = 3f(x)$	$(6, -12)$	$(1, 0)$
$y = f\left(\frac{x}{4}\right)$	$(24, -4)$	$(4, 0)$
$y = \frac{1}{5}f(x)$	$(6, -0.8)$	$(1, 0)$
$y = -f(x)$	$(6, 4)$	$(1, 0)$
$y = f(-x)$	$(-6, -4)$	$(-1, 0)$

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = 2f(x) + 3$ $y = 3f(x) - 2$	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = 5f(x) - 6$ $(3, 14)$

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation:
$y = f(2x) + 3$	$y = f(5x) - 6$
	$\left(\frac{3}{5}, -2\right)$
$y = f(3x) - 2$	

Worked example	Your turn
The point $A(2,5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = -f(x) + 3$ $y = -f(x) - 2$	The point $A(3,4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = -f(x) - 6$ $(3, -10)$

Worked example	Your turn
The point $A(2,5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = f(-x) + 3$ $y = f(-x) - 2$	The point $A(3,4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = -f(-x) - 6$ $(-3, -10)$

Worked example	Your turn
The point $A(2,5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = -2f(x) + 3$ $y = -3f(x) - 2$	The point $A(3,4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = -5f(x) - 6$ $(3, -26)$

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = 2f(-x) + 3$ $y = 3f(-x) - 2$	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = 5f(-x) - 6$ $(-3, 14)$

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation:
$y = -2f(-x) + 3$	$y = -5f(-x) - 6$
	$(-3, -26)$
$y = -3f(-x) - 2$	

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = 3f(2x) + 7$ $y = 7f(5x) - 2$	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = 5f(3x) - 7$ $(1, 13)$

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = -3f(2x) + 7$ $y = -7f(5x) - 2$	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = -5f(3x) - 7$ $(1, -27)$

Worked example	Your turn
The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve: $y = -3f(-2x) + 7$ $y = -7f(-5x) - 2$	The point $A(3, 4)$ is on the graph of $y = f(x)$. Write the new coordinates of A after the transformation: $y = -5f(-3x) - 7$ $(-1, -27)$